Macroeconomics Sequence Summary Notes[1]

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[1] Camilo Granados (cagranados8@gmail.com): these notes may contain typos and errors. I appreciate any feedback on them.

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Part I

Building Blocks of Monetary Policy Models

Context of macro models:

New Classical Synthesis:
IS-LM framework ⇒ AD curve
AS curve defined by LR AS (Perf. Inelastic/Natural rate hyp. - Classics), SR-AS (Perf. Elastic/Nominal Rigidities - Keynesians).
Adjustment (SR to LR): $\pi = \alpha(y - y^n) / \ \pi = -\beta(u - u^n)$

Breakdown: Stagflation (70's)
Friedman (1968) critique: Adj. of prices is not consistent with natural rate hyp. ⇒ Exp. Aug. PC: $\pi = \alpha(y - y^n) + E\pi$ (Include Expectations)

Lucas (1973): Need to include feedback of policies with expectations (Rational expectations), Microfounded models instead of reduced form models (agents' understanding of structural relations in the economy was more important than deducting expectations from past observations (adaptative). Identification is key)
Split of schools: NC focused on stochastic properties of Ramsey model (RBC), NK started to study nature of imperfections, Growth theorists: explored extensions of Ramsey NGM.

New-New classical synthesis: RBC model + w,p Rigidities (New Keynesians)
B.Cycles are caused only by real shocks but nominal rigidities lead to inef. outcomes ⇒ role for policy.
Lags of w,p included in models ⇒ stabilizing role of monetary policy

Financial Crisis:
Financial Frictions (from 2008) - Unconventional monetary policy Financial Regulation Debt management - Fiscal multiplier

General picture:

- Technological progress goes through waves (medium-long term) ⇒ they alone don't explain fluctuations.
- Expectations affect AD contemporaneously
- b/c of nominal rigidities AD affects AS in the SR.

  This leads to anticipations playing a big role:

- AD: Output is demand-determined. AD depends on anticipations of future output and interest rates.
- PC: Inflation depends on output and anticipations of future inflation.
- MP: Mon. Pol. affects real interest rates (due to nom. rigidities).

This led to a synthesis:

NK model:

RBC + two imperfections: Monopolistic Competition (goods markets), Partially sticky prices (Calvo).

Equations:

AD: Derived from FOC of HH, gives Output as function of real interest rate and future expected output.
PC: given Calvo pricing gives Inflation as f/n of expected inflation and output gap.
MP: Taylor R., i.e., interest rate as f/n of inflation and output gap (money demand is not explicit anymore).

Tools: Log-normal distribution: X ∼ lognormal if Z $\sim N(\mu, \sigma^2)$ with $Z = log(X)$
Property: $E[exp(Z)] = exp(\mu + \frac{1}{2}\sigma^2)$
X,Y jointly normal, then $E[X|Y] = \mu_x + \frac{\sigma_{12}}{\sigma_2^2}(y - \mu_y)$

Taylor Expansion: $f(x) = f(x_0) + f'(x_0)(x - x_0) + \sum_{n=2}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$

1 Modelling the Aggregate Supply

Modelling AS:

Table 1: Approaches to model AS

		Do markets clear	
		Yes	No
Is monetary policy neutral	Yes	**Classical RBC** (Kydland and Prescott)	**Real rigidity** (e.g.) efficiency wage (Akerlof, Yellen)
	No	**Imperfect information** (Friedman, Lucas '77)	**Nominal rigidity** Nominal contracts, menu costs (Fischer, Taylor, Calvo)

1.1 Imperfect information approaches

1.1.1 Lucas Island Model

(Imperfect Information)

Q: How to model that money affects output with Rational Expectations (RE)
Goal: Preserve frictionless market assumption but allow $\Delta M \rightarrow \Delta Y$ Implication: Sargent-Wallace policy irrelevance result

- J islands each produces a different product j
- Supply: f/n of relative prices $y_j \approx p_j - P$
- **Information:** p_j obs only by j, P unobserved $\Rightarrow y_j = \gamma(p_j - E[P|I_j])$
- Shocks: (Note: $corr(z_j, P) = 0, corr(p_j, P) \neq 0$)
  - General monetary shock $\tilde{P} \sim N(\mu_t, \sigma^2)$
  - Factor specific shock $\tilde{z}_j \sim N(0, \tau^2)$

$\boxed{p_j \equiv \tilde{P} + \tilde{z}}$ $\Rightarrow$ if y_j reacts to p_j monetary policy can affect output (since type of shock is not discernible).

To choose y_j producers need $E[P|I_j]$, consider a linear approach:

$$p_t = \alpha + \beta p_{jt} + \varepsilon_t \Rightarrow \hat{p}_t = \hat{\alpha} + \hat{\beta} p_{jt} = E[P_t|I_{jt}]$$

$$\hat{\beta} = \frac{Cov(p_{jt},P)}{var(p_{jt})} = \frac{Cov(\tilde{Z}_j+P,P)}{Var(\tilde{Z}_j+P)} = \frac{\cancelto{0}{Cov(\tilde{z}_j,P)}+Var(P)}{Var(\tilde{Z}_j)+Var(P)} = \frac{\sigma^2}{\sigma^2+\tau^2}$$

(assume $\alpha = 0$) $\Rightarrow y_j = \gamma(p_j - E[P|I_j]) = \gamma(p_j - \hat{\beta} p_j) = \gamma \frac{\tau^2}{\sigma^2+\tau^2} p_j$.

(key) Signal to noise ratio: $\frac{\tau^2}{\sigma^2}$ (0: all noise, high: Reaction of output to shocks in p_j, reaction is given by $\hat{\beta}$).

AS: $Y = \int y_j d_j = \gamma \frac{\tau^2}{\sigma^2+\tau^2} \int p_j dj \longrightarrow Y = \theta P$
$\boxed{Y_t = \gamma \frac{\tau^2}{\sigma^2+\tau^2}(P_t - E_{t-1}[P_t])}$ (Lucas supply function)

1.1.2 Sargent - Wallace policy irrelevance result

AD: $\underbrace{m_t + v_t}_{\text{policy + demand shocks}} = \underbrace{p_t + y_t}_{\text{Nominal GDP}}$

AS: $y_t = \theta(p_t - E_{t-1}[p_t])$

$$
\begin{aligned}
E_{t-1}P_t &= E_{t-1}(m_t + v_t - y_t) && \text{From (AD):}\\
y_t &= \theta(p_t - E_{t-1}(m_t + v_t - y_t)) && \text{plug in (AS):}\\
y_t &= \theta(m_t + v_t - y_t) - \theta E_{t-1}(m_t + v_t - y_t) && \text{replace } p_t\text{:}
\end{aligned}
$$

solve for y_t:

$$(1+\theta)y_t - \cancelto{0}{\theta E_{t-1}y_t} = \theta(m_t + v_t - E_{t-1}(m_t + v_t))$$

$\boxed{y_t = \frac{\theta}{1+\theta}\left[(m_t - E_{t-1}m_t) + (v_t - E_{t-1}v_t)\right]}$ (Output changes only due to unexpected monetary policy or demand).

Suppose policy is: $m_t = \delta Q_{t-1} + \varepsilon_t$ (systematic policy + policy error).
Under RE: $E_{t-1}m_t = \delta Q_{t-1}$ then: $y_t = \frac{\theta}{1+\theta}(\varepsilon_t + v_t - E_{t-1}v_t)$ (systematic policy rules are irrelevant, only shocks matter).

In AS: Slope $\propto \sigma^2$ then if MP is stable (σ^2 small) then AS flatter and AD shocks have larger effects on prices.

Lucas cross regime text: $x = p + y$ (nom. gdp.), take Δx as proxy of demand shocks. Compare 18 countries with different $\sigma^2_{\Delta x}$, look how real output reacts to shocks: $\Delta y \approx \beta \Delta x$.
Finding: Countries with larger $\sigma^2_{\Delta x}$ have smaller β (steeper AS) as expected.

Critiques: Corr $\neq$ Causation, omitted variables, e.g. due to more volatile π firms don't sign long term contracs or with higher π firms may be willing to assume menu costs (in both cases there's less nominal rigidity).

Consistency with optimizing behaviour is not explicit. Elasticity of supply to prices is huge (not plausible, CPI changes monthly and by a little amount).

1.2 Non-Market Clearing approaches

1.2.1 Fischer ('77), Taylor - Contract models

RE implies policy irrelevance by continuous market clearing via contracts that allowed for flexible prices.

Claim: If wage contracts are long enough a determininistic MP can have stabilizing effects. How: Nominal contracts that last longer than the time MP reacts to the economy state. Finding: SW assumptions (RE,MP rule based on lagged variables) are necessary but not sufficient for irrelevance.

Simple contract model (pre-Fischer)

$w_t = E_{t-1}p_t$ (1) [one period wage stickyness]

$y_t = -(w_t - p_t)$ (2) [AS]

$y_t + p_t = m_t - \tilde{v}_t$ (3) [AD]

$\tilde{v}_t = D(L)v_{t-1} + \eta_t$, $\eta_t \sim WN$ (4)

$m_t = b(L)v_{t-1}$ (5) [Deterministic MP rule]

subs (1) in (2): $y_t = -(E_{t-1}p_t - p_t)$ (6)
subs (6) in (3): $-(E_{t-1}p_t - p_t) + p_t = m_t - \tilde{v}_t \rightarrow p_t = \frac{1}{2}(m_t + \tilde{v}_t) + \frac{1}{2}E_{t-1}p_t$ (7)
take $E_{t-1}[\cdot]$: $E_{t-1}p_t = (b(L) + D(L))v_{t-1}$ (8)
(8),(7) in (6): $\mathbf{y_t = \frac{\eta_t}{2}}$ (only surprises matter)

Explanation: $(y = f(\eta))$ rational agents set wages at $t-1$, they already observe v_{t-1} so predict v_t, p_t undoing serial correlation, leaving only η_t.
Notice with no nominal friction: $w_t = p_t \rightarrow y_t = 0$ (flexible price)

Fischer two-period staggered contracts

- 1/2 population sets wages in odd periods, 1/2 in even.
- each contract lasts two periods

$w_t^i = E_{t-i}p_t$ for $i = 1, 2$ (1) wages
$y_t = \frac{1}{2}(w_t^1 - p_t) - \frac{1}{2}(w_t^2 - p_t)$ (2) AS
$y_t + p_t = m_t - v_t$ (3) AD
$m_t = b(L)v_{t-1}$ (4) MP

(1) → (2): $y_t = -\frac{1}{2}(E_{t-1}p_t - p_t) - \frac{1}{2}(E_{t-2}p_t - p_t)$ (5)
(2) → (3): $2p_t = m_t + v_t + \frac{1}{2}E_{t-1}p_t + \frac{1}{2}E_{t-2}p_t$
take E_{t-1} and E_{t-2}:
$E_{t-1}p_t = E_{t-1}m_t + E_{t-1}v_t$
$E_{t-2}p_t = E_{t-2}m_t + E_{t-2}v_t$
subs in (5) after evaluating expectations $(E_{t-1}m_t = m_t, E_{t-2}m_t, E_{t-1}v_t = D(L)v_{t-1}, E_{t-2}v_t = D(L)v_{t-2})$:
$\mathbf{y_t = \frac{1}{4}(m_t - E_{t-2}m_t) + \frac{1}{2}\eta_t + \frac{1}{4}\sum_{i=2}^{\infty}\eta_{t-i}}$

How to set MP?: compare $m_t = \bar{m}$ and $m_t = -D_1\eta_{t-1}$ in once case variance can be minimized with respect to a constant. Therefore MP can have an stabilizing effect
Conditions: Central bank has information advantage (knows η_{t-1} before half of agents). Given wage setting scheme sharing that information wouldn't change the outcome (unlike Lucas Island model). Extra info should be relevant.

Indexation: If only real wage matters why not to index it to p_t?

Question (Gray): Why contracts may not be fully indexed

1.2.2 Gray Indexing model

$p + y = m + v = \tilde{\mu}$ (1) AD
$y = \beta L + \tilde{\varepsilon}$ (2) AS (w/ product. shock)
$L = -\alpha(w - p) + \alpha\tilde{\varepsilon}$ (3) Labor Demand
$w_t = (1-\theta)E_{t-1}p_t + \theta p_t$ (4) Wage rule

Wage contract is set a period in advance but degree of indexation θ can be chosen (if zero we are in the former case).
Aim: Choose optimal θ to min $Var(L)$

$\min Var(L)$ s.t. $Ew = Ep$
$Ey = 0$

$Ey = 0$ with $E\tilde{\mu} = 0$ implies in (AD) $Ep = 0 \Rightarrow w_t = \theta p_t$

in (LD): $L = -\alpha(\theta - 1)p + \alpha\tilde{\varepsilon}$
subs in (AS): $y = \alpha\beta(1-\theta)p + (1+\alpha\beta)\tilde{\varepsilon}$
in (AD): $p = \frac{1}{1+\alpha\beta(1-\theta)}[\tilde{\mu} - (1+\alpha\beta)\tilde{\varepsilon}]$

Note we have two shocks $(\tilde{\mu}, \tilde{\varepsilon})$ and only one instrument (θ):

if $\theta = 1$ p moves 1 to 1 with $\tilde{\mu} \rightarrow y = (1-\alpha\beta)$ but AS is perfectly elastic (90 degrees AS) (nom. shock adressed, but unable to addressed real shocks (MPL)).
$0 < \theta < 1$ and AS has ≈ 45 degrees slope but $L = \frac{\alpha}{1+\alpha\beta(1-\theta)}[(1-\theta)\tilde{\mu} + \theta\tilde{\varepsilon}]$, (Doesn't minimize variance)

Conclusion: Agents don't fully index because they face tradeoff b/w adjusting to nominal shocks and real shocks (MPL).

Critics: Contracs are not rational (ex-ante is better to take spot wages, ex-post there should be renegotiation after one period to adjust to new shocks), agents are not optimizing.
Response: Long term contracs are observed, possible explained by costs of renegotiation or long term relationships that make firms not adjust to ST LD curve.

(More importantly) **real wage is countercyclical** in Fischer model: (+AD shock) $\uparrow p$, same w then $\downarrow$ real wage $\rightarrow \uparrow L \Rightarrow y$ (disagrees with data).

How to make $\frac{w}{p}$ procyclical?:

- Drop competitive markets ($p = MC$) for Mark-up scheme ($p = \mu MPL$)
- Include real rigidities: p, w sticky (lags in adjustment). Still effects of MP should go beyond contracts.
- Add technology ($\uparrow y$ but $\uparrow$ MPL via efficiency).

1.2.3 Taylor pre-fixed prices model

- persistent effects of MP: Staggered adjustment of firms' prices (firm's adjust prices just once per year)
- w, p fixed within contract
- Continuum of firms [0,1]

$y + p = \tilde{m}_t$ (1) AD
$p^*_{i,t} = m_t$ (2) Optimal price for firm i
$p_{i,t} = \int_{s=0}^{1} E_t p^*_{i,t+s} ds = \int_{s=0}^{1} E_t m_{t+s} ds$ (3) Stagg. adjustment of firms
$\int_{i=0}^{1} p_{i,t} di = \int\int E_t m_{t+s} ds di$ (4) Aggregate price

i.e. $p_t \propto E_t m_{t+s}$ and since $y = m - p$ then output depends on unexpected MP.

Improvement but MP effects are short $\Rightarrow$ Add Stragegic Complementarity (firms care about relative price).

$p^*_{i,t} = \theta m_t + (1-\theta)p_t$ (prices now take more time to adjust)
(Fixed length (prefixed contracts) + Stagg. adjustment with Strategic Complementarity)

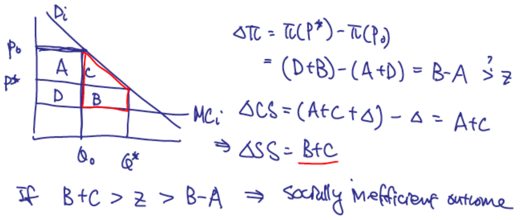
1.2.4 Menu Cost Model (Mankiw)

Firms choose not to change p_i in some neighborhood $[p^*_i \pm z]$ where z is a menu cost.

Set up:

N Firms have some monopolistic power ($i = 1, \ldots, N$).
Individual p_i decisions have aggregate effects (externalities).

Partial eq: At $t = 0$ firm sets p_0, then $\downarrow m^S, \downarrow Dem$, then $\downarrow p^*$



Given a marginal cost MC_i the profit changes by B-A, facing menu cost Z firm will adjust prices only if $B - A > Z$.

Social loss (change in social surplus): B + C. If it is too large then there is an inefficient outcome.

<u>Claim:</u> Decision to change prices has 1st order affects on social welfare but only 2nd order effect in firms' profit.
(Proof) Taylor exp. of $\pi(p_0)$ about p^*: $\pi(p_0) \approx \pi(p^*) + \cancelto{0}{\pi'(p^*)}(p_0 - p^*) + \frac{1}{2}\pi''(p^*)(p_0 - p^*)^2$

$$\Delta\pi = \pi(p_0) - \pi(p^*) \propto (p_0 - p^*)^2 \text{ whereas } \Delta SW \propto (p_0 - p^*) + (p_0 - p^*)^2$$

(if Δp is small 1st order effect $>>$ 2nd order effect. Only the latter matters for firms so they won't be willing to adjust prices and social welfare would be hurt)

1.2.5 Blanchard - Kiyotaki

Firms exert <u>AD externalities</u> on each other.
Dixit-Stiglitz setup with differentiated goods:

- Differentiated goods
- Monopolistic competition
- N firms (large)

$y_i = D_i(\frac{p_i}{P}, \frac{M}{P})$

Continuum of HH- producers: [0,1]

$\max U(C_i, \frac{m_i}{P}, L_i) = (\frac{C_i}{\alpha})^{\alpha}(\frac{M_i/P}{1-\alpha})^{1-\alpha} - \frac{L_i^{\beta}}{\beta}$ (MIU).

$C_i = \left[\int_0^1 c_{ij}^{\frac{\sigma-1}{\sigma}} dj\right]^{\frac{\sigma}{\sigma-1}}$ (consumption over varieties).

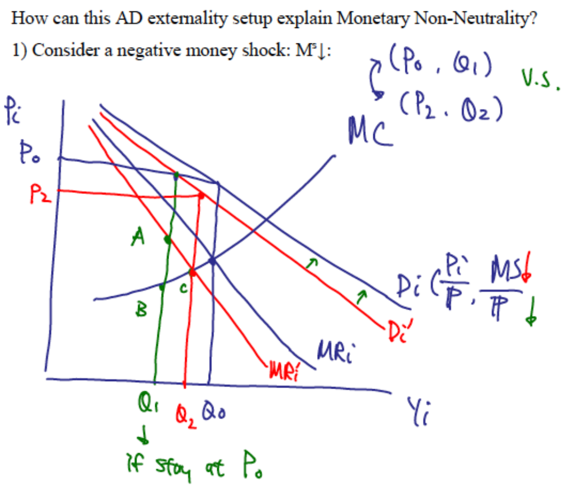
$p_i y_i = \int_0^1 p_j c_j dj + M_i$ (BC)

Production f/n: $y_i = L_i$

$\downarrow M^s$, decision: (p_0, Q_1) vs (p_2, Q_2) (latter is optimal w/o costs).

Without menu cost: at (p_0, Q_1) $MR > MC$ then $\uparrow Q \Rightarrow \downarrow p_i \quad \forall i \rightarrow \downarrow P \Rightarrow \frac{M^s}{P}$ goes back up $\rightarrow \uparrow D_i \rightarrow \uparrow Q_i \Rightarrow \uparrow Y$
Output returns to initial level.

Equilibrium: $\{y_i, p_i\}$ $s.t.$ $S_i = D_i$ $\forall i \in [0,1]$ and π_i is maximized. $\rightarrow P = \left[\int_0^1 p_j^{1-\sigma} dj\right]^{\frac{1}{1-\sigma}}$, $y_i = D(\frac{p_i}{P}, Y)$
(Y is the AD and $\frac{M}{P} = Y$)


How can this AD externality setup explain Monetary Non-Neutrality?

1) Consider a negative money shock: M^s↓:

(key: If prices are adjusted then output goes to $Q_2 > Q_1$ there are gains in profit given by $\triangle ABC$ and afterwards output returns no normal.)

With menu cost: If $Z > \triangle ABC$ firms may not change prices $\Rightarrow \downarrow Y$

Results:

Monetary non neutrality: $\Delta M^s \Rightarrow \Delta Y$ due to $\bar{p}_i$ (potential justification for monetary expansion).
Pro-cyclical real wage: $(y_i = L_i) \downarrow Y \to\downarrow L^D \to\downarrow w, \bar{P} \Rightarrow\downarrow \frac{w}{P}$.
Large shocks are always socially costly as it may force firms to pay the menu cost.

Ball and Romer's critique:

L^s **is inelastic** ($\downarrow M^s \to \bar{P} \to\downarrow Y \to\downarrow L^D \Rightarrow\downarrow w \Rightarrow\downarrow MC \Rightarrow \triangle ABC$ is larger $\therefore Z$ must be very large not to change p_i)

Reply: that assumes Labor market clears which is not always the case $\Rightarrow$ in SR add L rididities (real rigidities, real wage sticky in the SR).

Possible causes: efficiency wages, unions, search-matching process, long term contracts.

1.3 Real rigidities approach

Efficiency wages (EW): Nutrition story (pay to gain in productivity), moral-hazard problem.

1.3.1 Shapiro-Stiglitz

Micro foundations for real wage rigidity via EW + Imperfect monitoring.

Set up:
- Firms face monitoring cost (cannot observe effort).
- at t workers choose effort e_t (either $\bar{e} > 0$ or 0)

Workers Utility: $u_t = w_t - e_t$ (if employed), $u_t = 0$ (if unemployed)
UMP: $\max U = \int_{t=0}^{\infty} e^{-\rho t} u_t(w_t, e_t)dt$

Three states at t:	Transition probabilities:
- Not Shirking (NS): $e_t = \bar{e} \longrightarrow V_E^{NS}$	$NS \to U : b$ (exog)
- Shirking (S): $e_t = 0 \longrightarrow V_E^{S}$	$S \to U : b + q$ (exog)
- Unemployed $\longrightarrow V_U$	$U \to (NS, S) : a$ (endog)

Firms PMP: $\max_{w_t} \pi = F(\bar{e}L_t) - w_t(L_t + S_t)$ (L_t is the quantity of non-shirking and S_t that of shirkers)

Asset eq: $V_t =$ contemp. payoff + expected discounted continuation payoff ($\rho E_t V_{t+1}$)

consider a small unit of time $d_t = [0, t]$ and approximate discount factor as $e^{-\rho t} \approx 1 - \rho t$

$$V_E^{NS} = (w - \bar{e})t + (1 - \delta t)[btV_U + (1 - bt)V_E^{NS}] \quad \Rightarrow \quad V_E^{NS} = \frac{(w-\bar{e})t+(1-\rho t)btV_U}{1-(1-\rho t)(1-bt)} \overset{t\to 0}{\to} \frac{0}{0}$$

$\xrightarrow{\text{L'Hop}} V_E^{NS} = \frac{(w-\bar{e})+bV_U}{\rho+b}$ (eval. at $t = 0$). Solving analogously in each case:

$$\begin{aligned} \rho V_E^{NS} &= (w - \bar{e}) + b(V_U - V_E^{NS}) \quad &(1) \\ \rho V_E^{S} &= w + (b + q)(V_U - V_E^{S}) \quad &(2) \\ \rho V^{U} &= 0 + a(V_E - V_U) \quad &(3) \end{aligned}$$

In an equilibrium implementing No Shirking we have: $V^{NS} = V^S = V_E$, i.e. (1)=(2):

$w - \bar{e} + b(V_U - V_E) = w + (b + q)(V_U - V_E) \quad \Rightarrow \quad V_E - V_U = \frac{e}{q}$
then from (1) and (3): $\rho\frac{\bar{e}}{q} = w - \bar{e} - (a + b)\frac{\bar{e}}{q} \quad \Rightarrow \quad \mathbf{w} = \bar{\mathbf{e}} + \frac{\bar{\mathbf{e}}}{\mathbf{q}}(\mathbf{a} + \mathbf{b} + \boldsymbol{\rho})$

wage is increasing in the effort ($\bar{e}$), the probatility of finding a job (a), the probability of break up (b) and decreasing in the probatility of detecting shirkers (q).

To put wage in terms of labor force: In SS,

job creation = job destruction
$a(\bar{L} - NL) = (NL)b \quad \Rightarrow \quad a = \frac{NL}{\bar{L}-NL}b \to a + b = \frac{L}{\bar{L}-NL}b$ then: $\mathbf{w} = \bar{\mathbf{e}} + (\boldsymbol{\rho} + \frac{\mathbf{L}}{\bar{\mathbf{L}}-\mathbf{NL}}\mathbf{b})\frac{\bar{\mathbf{e}}}{\mathbf{q}}$

Wage is increasing in the employment (N: number of firms, L: employed by firm). At full employment ($\bar{L} = NL$) unemployed workers find work instantly and there is no cost of being fired then no wage can deter shirking.

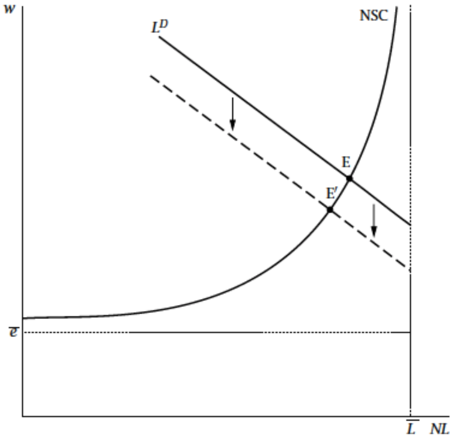


Figure 1: No Shirking Condition: Drop in L^D

Notice that $\Delta w^{NSC} < \Delta w$ implied by inelastic labor supply.

Other shocks:

Increase in monitoring quality ($\uparrow q$): Makes NSC curve closer to Labor Supply curve.

No turnover ($b = 0$): unemployed are never hired, base wage increases to $\bar{e} + \rho\frac{\bar{e}}{q}$ this is also the NSC wage.

In the SR firms hire based in NSC
In the LR monitoring is perfect and firms hire based on L^S ($w = \bar{e}$).
here w is real wage.

(Monetary policy) Combine Blanchard-Kiyotaki menu cost model to justify MP non neutrality:
$\downarrow M^S \to \bar{p}_i$ due to $z \Rightarrow \bar{P} \to \downarrow Y$ ($\downarrow L^D$ due to NSC that prevents complete wage adjustment $\downarrow w$)

1.4 Nominal Rigidities

1.4.1 Price Rigidities: State-Contingent scheme

Capling and Spulber '87

Endogenous price adjustment ([s,S] scheme, elevator model)

Set up:

- Heterogeneous firms facing fixed menu cost.

- Initial distribution of prices uniform $[M - s, M + s]$.

- On aggregate P = M and $P_i^* = M$ but due to menu costs firm may not price at P_i^* if $P_i \in [M - s, M + s]$

- Upon ΔM only firms dropped out of the interval would find worthwhile to adjust prices. They adjust depending on $E[M]$ and not on current M. If a permanent increase of M is expected then in anticipation, to avoid incurring in more menu costs, firms move to the end of the interval. They preserve the same uniform distribution around the new M $\Rightarrow P = M$ still and $Y = M - P = 0$, i.e. prices adjust to MP at once.

Problems:

- CS doesn't generalize state-contingent models.

-One sided MP framework ($\uparrow M$).

- Menu cost is fixed: Infrequent larger adjustments. If convex adjustment cost are considered then there are frequent small changes.

- Firms are heterogeneous but face the same common shock (not consistent in terms of signal to noise, e.g. Lucas Island model).

Then there is NO relationship between infrequent price adjustment and dynamic response of P to MP. Some price stickyness is regained only by relaxing assumptions.

Findings:

- price is three times as flexible as the frequence of microeconomic price adjustment (many firms don't move but aggregate price is still flexible).

- Strategic complementarities may reduce flexibility but less than in Calvo type models.

1.4.2 Time-contingent scheme (Calvo and Yun)

Calvo (1993) and Yun (1996)

NK-PC microfounded, firms adjust prices randomly in staggered fashion. Price stickyness is costly (friction of the model). Alternative: Rotemberg (82) slow price adjustment due to convex costs.

Prob. of changing price at t: (1-θ)

Expected duration:

$E[t] = \sum tPr(x \leq t) = (1-\theta) + 2\theta(1-\theta) + 3\theta^2(1-\theta) + \dots = 1 - \theta + 2\theta - 2\theta^2 + 3\theta^2 - 3\theta^3 + \dots = \frac{1}{1-\theta}$

Price output trade off:

$p^*_{t,j} = p_t + \phi y_t$ (same as Taylor $p^*_{t,i} = \lambda M_t + (1-\lambda)P_t$ substituting $M_t - P_t = y_t$ then ϕ **means real rigidity**[2]).

Profit loss in period t from setting a price x_t is:

$\pi(p^*_t) - \pi(x_t) = \cancelto{0}{\pi'(p^*_t)}(p^*_t - x_t) + \pi''(p^*_t)(p^*_t - x_t)^2 \Rightarrow \frac{\kappa}{2}(p^*_t - x_t)^2$ (loss function)

Problem of the firm: (minimize expected loss)

$\min_{x_t} \sum_{j=0}^{\infty}(1-\theta)\theta^j\beta^i\frac{\kappa}{2}(E_tP^*_{t+j} - x_t)^2$

FOC: $\kappa(1-\theta)\sum_{j=0}^{\infty}\theta^j\beta^j(E_tp^*_{t+j} - x_t) \quad \Rightarrow \quad x_t = (1-\theta\beta)\sum_{j=0}^{\infty}\theta^j\beta^jE_tp^*_{t+j}$

forward x_t to $t+1$, fix the sum indexes so it starts at $j = 1$, multiply times $\theta\beta$, take expectations ($E_t(\cdot)$), and rearrange:

$(\theta\beta)E_tx_{t+1} = (1-\theta\beta)\sum_{j=1}^{\infty}\theta^j\beta^jE_tp^*_{t+j}$

rearrange x_t:

$x_t = (1-\theta\beta)\theta^0\beta^0E_tp^*_t + (1-\theta\beta)\sum_{j=1}^{\infty}\theta^j\beta^jE_tp^*_{t+j} = (1-\beta\theta)p^*_t + \beta\theta E_tx_{t+1}$

replace $p^*_t = p_t + \phi y_t$:

---

[2] $p^*_{t,i} = \lambda(y_t + P_t) + (1-\lambda)P_t = P_t + \lambda y_t$ then $\phi = \lambda$.

$x_t = (\theta\beta)E_t x_{t+1} + (1-\theta\beta)p_t + (1-\theta\beta)\phi y_t$

rearrange to get: $x_t - p_t - \theta\beta E_t x_{t+1} = -\theta\beta p_t + (1-\theta\beta)\phi y_t$

$x_t - p_t - \theta\beta E_t x_{t+1} + \theta\beta\mathbf{E_t p_{t+1}} = \theta\beta\mathbf{E_t p_{t+1}} - \theta\beta p_t + (1-\theta\beta)\phi y_t$
$z_t - \theta\beta E_t z_{t+1} = \theta\beta\pi_{t+1} + (1-\theta\beta)\phi y_t \quad (\star)$

aggregrate prices are also given by:

$p_t = (1-\theta)\sum_{j=0}^{\infty}\theta^j x_{t-j} = (1-\theta)x_t + \theta(1-\theta)\sum_{j=0}^{\infty}\theta^j x_{t-1-j}$

i.e. $p_t = (1-\theta)x_t + \theta p_{t-1}$

get an expression for the inflation:

$(1-\theta)p_t + \theta p_t = (1-\theta)x_t + \theta p_{t-1}$
$\theta(p_t - p_{t-1}) = (1-\theta)(x_t - p_t)$
$\theta\pi_t = (1-\theta)z_t$

forward, take E_t and multiply by $-\theta\beta$, getting $-\beta\theta^2 E_t\pi_{t+1} = -\beta\theta(1-\theta)E_t z_{t+1}$, add it to the inflation eq:

$(1-\theta)(z_t - \beta\theta E_t z_{t+1}) = \theta(\pi_t - \beta\theta E_t\pi_{t+1}) \quad (\clubsuit)$

subs. $(\star)$ in the LHS of $(\clubsuit)$:

$(1-\theta)(\theta\beta\pi_{t+1} + (1-\theta\beta)\phi y_t) = \theta(\pi_t - \beta\theta E_t\pi_{t+1})$
$\theta\beta E_t\pi_{t+1} + (1-\theta)(1-\theta\beta)\phi y_t = \theta\pi$
$\pi = \beta E_t\pi_{t+1} + \frac{(1-\theta)(1-\theta\beta)}{\theta}\phi y_t$

finally $\boxed{\pi_\mathbf{t} = \beta\mathbf{E_t}\pi_\mathbf{t+1} + \lambda\mathbf{y_t}}$ with $\lambda = \frac{(1-\theta)(1-\theta\beta)}{\theta}\phi$

λ: output-inflation trade-off (the smaller, the larger real effect of inflation on output).

$\downarrow\lambda$ when:
- $\downarrow\phi$ i.e. more real rigidity (prices are not adjusted by (real) money, think of Taylor strat. compl. pricing eq.)
- $\uparrow\theta$ i.e. more nominal ridigity, prices become too unlikely to change.

1.5 Financial Frictions

Credit frictions channel of Monetary Policy

Credit Channel of MP: Imperfect info and other frictions in credit markets.

(Context) Miller-Modigliani (1958) theorem: Capital structure irrelevance, i.e., value of firms doesn't depend on how it's financed, i.e., Debt/Equity ratio, or leverage won't matter.

- This result does not hold with other imperfections, e.g. asymetric information, agency costs and the consequent external finance premium (EFP).

- Explanation: In real sector (AD) investment is sensitive to financial variables (net worth, cash flow). This phenomenom leads to an amplification of the transmission of MP to the investment and output.

- MP should affect only SR FFR and not LR rates, however it impacts purchases of long lived assets and therefore affects the magnitude, timing and composition of MP transmission.

Standard MP transmission channel: $\downarrow m \rightarrow\uparrow i \rightarrow\downarrow (I, C, ER) \Rightarrow\downarrow y$

Amplification: $\downarrow m \to \downarrow (\pi^e, \text{cash flow}) \to \Delta$borrowers balance sheet + asym. info (Adv.S., MH) $\Rightarrow \downarrow$ Lending $\to \downarrow I \Rightarrow \downarrow y$.

Policy implications: Justification for financial stability policies (capital requirements, FX reserve requirements among others.)

1.5.1 Bernanke, Gertler and Gilchrist (1999)

Costly state verification model (principal-agent type).

$t = 0, 1$

$QK = N + B$ (Q: Price of capital, K: Capital, N: Net worth, B: Debt)

$\tilde{w}R_kQK$ (project payoff $E(\tilde{w}) = 1, \tilde{\in}[\underline{w}, \bar{w}], H(\tilde{w}) = P(\tilde{w} < w), h(\tilde{w}) = \frac{dH}{dw}, R_k$: Avg. gross return of capital)

First Best: (symmetric info) Entrepreneur operates if $R_k > R$ (R: Oportunity cost), by perfect competition: $R_k = R$

then the benchmark case bears no friction: Investment decisions is independent of financial structure of firms.

Second Best: (private info and limited liability)

Lenders pay a cost of monitoring, a fraction of return: $\mu\tilde{w}R_kQK$

LL: Entrepr. payoff bounded at zero → agent has incentives to misreport returns.

Agent UMP: Choice of K
Constraint: Lender receives oportunity cost in expectation.
Contract: Payment based on w and decision to monitor.

Optimal contract: Induced truth-telling, minimizes expected monitoring costs.

$D = w^*R_kQK$ where D: face value of debt, w^*: cutoff value of w and w: obs. return multiplier.

$$\text{if } w \geq w^* = \begin{cases} \text{lender:} & D = w^*R_kQK \\ \text{borrower:} & (w - w^*)R_kQK \end{cases} \qquad \text{if } w < w^* = \begin{cases} \text{lender:} & (1-\mu)wR_kQK \\ \text{borrower:} & 0 \end{cases}$$

$\therefore$ deadweight bankrupcy cost: $\mu w R_kQK$
in OC:

- No incentive to lie (non default states payment for lender is fixed, O.W. there is monitoring).
- Non Default state D is minimized by giving the lender everything in default states.
- Given $D = w^*R_kQK$ bankrupcy prob.: $H(w^*) = H(\frac{D}{R_kQK})$ (increasing in D).

Lender expected gross payoff: $\int_{w^*}^{\bar{w}} w^*R_kQKdH + \int_{\underline{w}}^{w^*} wR_kQKdH \equiv \Gamma(w^*)R_kQK$
Lender expected net payoff: $\int_{w^*}^{\bar{w}} w^*R_kQKdH + \int_{\underline{w}}^{w^*} (\mathbf{1} - \boldsymbol{\mu})wR_kQKdH \equiv [\Gamma(w^*) - \mu G(w^*)]\, R_kQK$

$$\text{Entrepreneur UMP:} \quad \max_{w^*, K} \{[1 - \Gamma(w^*)]R_kQK - RN, 0\}$$
$$s.t. \quad [\Gamma(w^*) - \mu G(w^*)]\, R_kQK = R\underbrace{(QK - N)}_{B}$$

$\mathcal{L} = [1-\Gamma(w^*)]R_kQK - RN + \lambda([\Gamma(w^*) - \mu G(w^*)]\, R_kQK - R(QK-N))$

FOC:

$[w^*] : \Gamma'(w^*) = \lambda(\Gamma'(w^*))$

$[k] : ([1-\Gamma(w^*)] + \lambda[\Gamma(w^*) - \mu G(w^*)])R_k = \lambda R \longrightarrow R_k - \frac{\lambda}{[1-\Gamma(w^*)]+\lambda[\Gamma(w^*)-\mu G(w^*)]}R = 0$

$\Rightarrow R_k - \chi(w^*)R = 0$ with $\chi(w^*) > 1, \chi'(w^*) > 0$

$[\lambda] : [\Gamma(w^*) - \mu G(w^*)]\, R_k = R(1 - \frac{N}{QK})$ **(Lender's participation constraint)**

from $[\lambda]$: $\frac{QK}{N} = \frac{1}{1-[\Gamma(w^*)-\mu G(w^*)]\frac{R_k}{R}} \Rightarrow \boxed{\mathbf{\frac{QK}{N}} = \phi\left(\mathbf{\frac{R_k}{R}}\right)}$ with $\phi'(\cdot) > 0$

Then the Optimal leverage ratio is a positive f/n of the EFP (spread).

Aggregrating: $Q\bar{K} = \phi(\frac{R_k}{R})\bar{N}$

Relationship between balance sheet strength and spread (EFP):

$\frac{Q\bar{K}}{\bar{N}} = \phi\left(\frac{R_k}{R}\right) \Rightarrow \frac{R_k}{R} = \varphi\left(\frac{Q\bar{K}}{\bar{N}}\right)$ with $\varphi'(\cdot) > 0$ (Spread is a decreasing function of balance sheet strength).

Bottom line: *EFP*, due to agency costs, becomes **countercyclical**.

1.6 Monopolistic Competition (methodology)

- Dixit-Stiglitz preferences + MIU
- HH in $[0,1]$ each producing differentiated good (imperfect substitutes)
- Each HH consumes $C_i = \left[\int_0^1 C_{ij}^{\frac{\sigma-1}{\sigma}}\, dj\right]^{\frac{\sigma}{\sigma-1}}$ (love for variety)

Preferences: $U(C_i, \frac{M_i}{P}, L_i) = \left(\frac{C_i}{\alpha}\right)^{\alpha}\left(\frac{M_i/P}{1-\alpha} - \frac{L_i^{\beta}}{\beta}\right)$

with $C_i = \left[\int_0^1 C_{ij}^{\frac{\sigma-1}{\sigma}}\, dj\right]^{\frac{\sigma}{\sigma-1}}$, $P = \left[\int_0^1 P_j^{1-\sigma} dj\right]^{\frac{1}{1-\sigma}}$ (CES, $\sigma > 1$ is the elast. of substitution b/w goods j and k, if large they are close substitutes).

[BC]: $\int_0^1 P_jC_{ij}d_j + M_i = P_iY_i + \bar{M}_i$ ($\bar{M}_i$: Initial money holdings).

Technology: $Y_i = L_i$

Solution:

1. Solve for consumption demand for each good C_{ij}
2. Solve for HH allocation between consumption and money
3. Solve for production and pricing decisions (here is key to remember we assume homogeneous firms).

1. Consumption for each good: Two steps decision, how big is C_i and demand for each good given relative prices:

   HH maximizes consumption: $\max C_i = \left[\int_0^1 C_{ij}^{\frac{\sigma-1}{\sigma}} dj\right]^{\frac{\sigma}{\sigma-1}}$ s.t. $\int_0^1 P_j C_{ij} dj = X_i$ (X_i: HH spending)

   (given $\sigma > 1$) an analogous but simpler problem is:

   max $\left[\int_0^1 C_{ij}^{\frac{\sigma-1}{\sigma}} dj\right]$ s.t. $\int_0^1 P_j C_{ij} dj = X_i$

   $$L = \int_0^1 C_{ij}^{\frac{\sigma-1}{\sigma}} dj + \lambda \left(X_i - \int_0^1 P_j C_{ij} dj\right)$$

   FOC:

   $$[C_{ij}] : \qquad \tfrac{\sigma-1}{\sigma} C_{ij}^{\frac{-1}{\sigma}} = \lambda P_j \quad \Rightarrow \quad \frac{\mathbf{C_{ij}}}{\mathbf{C_{ik}}} = \left(\frac{\mathbf{P_j}}{\mathbf{P_k}}\right)^{-\sigma} \qquad (*)$$

   rewrite as: $\left(\frac{C_{ij}}{C_{ik}}\right)^{\frac{\sigma-1}{\sigma}} = \left(\frac{P_j}{P_k}\right)^{1-\sigma}$

   integrate over j: $\frac{C_i^{\frac{\sigma-1}{\sigma}}}{C_{ik}^{\frac{\sigma-1}{\sigma}}} = \frac{P^{1-\sigma}}{P_k^{1-\sigma}}$

   when integrating over j we used that from definitions of C_i and P_i: $C_i^{(\sigma-1)/\sigma} = \int_0^1 C_{ij}^{(\sigma-1)/\sigma} dj$ and $P^{1-\sigma} = \int_0^1 P_j^{1-\sigma} dj$

   then, $\mathbf{C_{ik}} = \left(\frac{\mathbf{P_k}}{\mathbf{P}}\right)^{-\sigma} \mathbf{C_i}$

   It follows (by multiplying by P_k and integrating over k) that $X_i = PC_i$

2. Allocation between C_i and M_i:

   With $PC_i = X_i$ rewrite original UMP as:

   $$\max \quad \left(\frac{C_i}{\alpha}\right)^{\alpha} \left(\frac{M_i/P}{1-\alpha}\right)^{\alpha} - \frac{L_i^{\beta}}{\beta} \quad \text{s.t. } \overbrace{C_i P}^{X_i} + M_i = P_i Y_i + \bar{M}_i$$

   (given $Y_i, \bar{M}_i$) the standard CD solutions are:

   $PC_i = \alpha(P_i Y_i + \bar{M}_i) \qquad M_i = (1-\alpha)(P_i Y_i + \bar{M}_i)$

   then, subs. in (+): $\quad C_{ij} = \frac{\alpha}{1-\alpha} \frac{M_i}{P} \left(\frac{P_j}{P}\right)^{-\sigma}$

3. Production and pricing decisions

   Replace C_i, M_i and $Y_i = L_i$ in the utility function:

   $$U = \left(\frac{\cancel{\alpha}}{P} \frac{(P_i Y_i + \bar{M}_i)}{\cancel{\alpha}}\right)^{\alpha} \left(\frac{1}{P} \cancel{\frac{1}{1-\alpha}} \cancel{(1-\alpha)} (P_i Y_i + \bar{M}_i)\right)^{1-\alpha} - \frac{Y_i^{\beta}}{\beta} \quad \Rightarrow \quad \boxed{\mathbf{U} = \frac{\mathbf{P_i Y_i}}{\mathbf{P}} - \frac{\mathbf{Y_i}^{\beta}}{\beta} + \frac{\mathbf{\bar{M}_i}}{\mathbf{P}}}$$

   Total demand for each good i is the sum over the k households demands:

   $$Y_i = \int_0^1 C_{ki} dk = \int_0^1 \frac{\alpha}{1-\alpha} \frac{M_k}{P} \left(\frac{P_i}{P}\right)^{-\sigma} dk = \frac{\alpha}{1-\alpha} \frac{1}{P} \left(\frac{P_i}{P}\right)^{-\sigma} \int_0^1 M_k dk \quad \Rightarrow \quad \boxed{\mathbf{Y_i} = \frac{\alpha}{\mathbf{1-\alpha}} \frac{\mathbf{M}}{\mathbf{P}} \left(\frac{\mathbf{P_i}}{\mathbf{P}}\right)^{-\sigma}}$$

thus we get the typical downward sloping AD curve $\left(Q_i = D_i\left(\frac{P_i}{P}, \frac{M}{P}\right)\right)$

Maximizing the new transformed $U(\cdot)$ gives (from the FOC):

$\frac{P_i}{P} = \frac{\sigma}{\sigma-1}Y_i^{\beta-1}$ **Price = markup x marginal cost**

replace Y_i:

$$\frac{P_i}{P} = \left[\frac{\sigma}{\sigma-1}\left(\frac{\alpha}{1-\alpha}\right)^{\beta-1}\left(\frac{M}{P}\right)^{\beta-1}\right]^{\frac{1}{1+\sigma(\beta-1)}}$$

$\therefore$ when ΔM the effect on Y and $\frac{P_i}{P}$ depends on β, σ. If $\beta = 1$ the relative price doesn't change and ΔM is accomodated by ΔY.

As $\beta > 1$, ΔM leads to $\Delta\frac{P_i}{P}$ together with quantity adjustment.

General equilibrium

With symmetric HH relative price is 1:

$$\boxed{1 = \frac{\sigma}{\sigma-1}Y^{\beta-1} \quad P = \frac{\alpha}{1-\alpha}\frac{M}{Y} \propto M}$$

Output is a constant markup and **money is neutral**, i.e., $\Delta M \Rightarrow \Delta P$ proportionally and the effect on Y is null.

with <u>menu cost</u> the models implies <u>monetary non neutrality</u>. Intuition: ΔM small then $\bar{P} \Rightarrow$ non neutrality. Again, deviations from optimal price have only second order effects on profits whereas first and second on social welfare.

Extra:

Monopolistic Competition - Supply Side Demand Derivation

(complementary, not included in the course)

Source: Chugh 2015, Modern Macroeconomics.

Retail firms:

$$\max_{\{y_{it}\}_{i=0}^{\infty}} P_t\left[y_t\right] - \int_0^1 P_{it}y_{it}di$$

$$\max_{\{y_{it}\}_{i=0}^{\infty}} P_t\left[\int_0^1 y_{it}^{1/\varepsilon}di\right]^{\varepsilon} - \int_0^1 P_{it}y_{it}di$$

F.O.C.:

$$[y_{it}]: \qquad \varepsilon P_t\left[\int_0^1 y_{it}^{1/\varepsilon}di\right]^{\varepsilon-1}\frac{1}{\varepsilon}y_{it}^{1/\varepsilon-1} - P_{it} = 0$$

Rearrange and substitute $y_t^{1/\varepsilon} = \int_0^1 y_{it}^{1/\varepsilon}y_{it}$ to obtain the firm's optimal demand function for the wholesale good:

$$y_{it}^{\frac{1-\varepsilon}{\varepsilon}} = \frac{P_{it}}{P_t}\left(y_t^{1/\varepsilon}\right)^{1-\varepsilon}$$

$$y_{it} = \left(\frac{P_{it}}{P_t}\right)^{\frac{\varepsilon}{1-\varepsilon}}y_t$$

The intuition is straightforward, the demand depends negatively on its relative price and positively on the aggregate demand.

Wholesale Firms

The i-th wholesale firm will set prices to maximize its profits:

$$\max_{P_{it}} \pi_{it} = P_{it}y_{it} - P_t mc_{it}y_{it}$$

Where mc_{it} is the real marginal cost of the i firm, P_t should be multiplied to put it in units of the final consumption good (or in nominal terms).

Substitute the monopolist optimal demand:

$$\begin{aligned}\max_{P_{it}} \pi_{it} &= P_{it}\left(\frac{P_{it}}{P_t}\right)^{\frac{\varepsilon}{1-\varepsilon}} y_t - P_t mc_{it}\left(\frac{P_{it}}{P_t}\right)^{\frac{\varepsilon}{1-\varepsilon}} y_t \\ &= P_{it}^{\frac{1}{1-\varepsilon}} P_t^{\frac{\varepsilon}{\varepsilon-1}} y_t - mc_t P_{it}^{\frac{\varepsilon}{1-\varepsilon}} P_t^{\frac{2\varepsilon-1}{\varepsilon-1}} y_t\end{aligned}$$

Where we assumed $mc_{it} = mc_t \forall i$.

F.O.C.:

$$[P_{it}]: \quad \frac{1}{1-\varepsilon} P_{it}^{\frac{\varepsilon}{1-\varepsilon}} P_t^{\frac{\varepsilon}{\varepsilon-1}} y_t - \frac{\varepsilon}{1-\varepsilon} P_{it}^{\frac{2\varepsilon-1}{1-\varepsilon}} P_t^{\frac{2\varepsilon-1}{\varepsilon-1}} y_t mc_t = 0$$

Rearranging to get rid of the constants and prices exponents:

$$\begin{aligned}P_{it}^{\frac{-\varepsilon}{1-\varepsilon}} P_{it}^{\frac{\varepsilon}{1-\varepsilon}} P_t^{\frac{\varepsilon}{\varepsilon-1}} - \varepsilon P_{it}^{-1} P_t^{\frac{2\varepsilon-1}{\varepsilon-1}} mc_t &= 0 \\ 1 - \varepsilon P_{it}^{-1} P_t mc_t &= 0 \\ P_{it} &= \varepsilon P_t mc_t\end{aligned}$$

In terms of relative prices ($p_{it} = \frac{P_{it}}{P_t}$):

$$p_{it} = \varepsilon mc_t$$

Other frequently used notation: $\varepsilon = \frac{\theta}{\theta-1}$, i.e., $p_{it} = \frac{\theta}{\theta-1} mc_t$

Key result: The profit maximizing price is a single, constant, mark-up over the marginal cost.

2 Consumption

Intro: Pre-RE context: Keynes, Friedman, Fisher.

(context) Pre-rational expectations: (Keynes, Fisher, Friedman) SR and LR relationship between Y and C, e.g. Lyfe-cycle hypothesis, Permanent Income hypothesis.

Keynes: ad-hoc linear relationship $C(Y) = a + bY$

- MPC less than 1 ($0 < b < 1$)
- Avg PC is decreasing in income: $APC = \frac{C}{Y} = \frac{a}{Y} + b$
- Interest rate doesn't play a role

In the SR all three claims seem to hold. In the LR a decreasing APC in income is no longer true.

Friedman/Modigliani/Fisher: (explanation) C shifts over time, people make consumption decision over life-time income (PI), then C,S is not a period by period decision.

Fisher:

Agent UMP is $\max_{\{c_t\}_{t=0}^T} (1+\rho)^{-t} u(c_t)$ s.t. $\sum_{t=0}^T (1+r)^{-t} Y_t + W_0 = \sum_{t=0}^T (1+r)^{-t} c_t$ (lifetime BC)

FOC: $[c_{t+j}]:$ $u'(c_{t+j}) = \lambda \left(\frac{1+\rho}{1+r}\right)^{t+j}$

if $r = \rho \quad \Rightarrow \quad u'(c_{t+j}) =$ constant $\forall t$

$\Rightarrow$ $c_t = c$, plug in BC and solve for consumption: $c = \frac{1}{\sum_{t=0}^{T}(1+r)^{-t}}\left[\sum_{t=0}^{T}(1+r)^{-t}Y_t + W_0\right]$
Then $\underline{\text{C decision is dynamic}}$ and depends on income over time and not over a single period.

Modigliani: Life-cycle pattern of income path

Friedman: PIH (Permanent vs Transitory income)

$C = C^P + C^T$
$Y = Y^P + Y^T$

Fisher's relation holds only for the permanent component, i.e., $C^P = Y^P$ (with $a = 0, b = 1$)

However, empirically only the sum is observed (Y,C). Then by estimating $C = \hat{a} + \hat{b}Y$ you get,

$\hat{b} = \frac{cov(C,Y)}{Var(Y)} = \frac{Cov(Y^P,C^P)}{Var(Y^P)+Var(Y^T)}$

Assume C^T, Y^T are white noise: $Cov(Y^T, C^P) = 0, Cov(C^T, C^P) = 0$, and $C^P = Y^P$ then,

$\hat{b} = \frac{Var(Y^P)}{Var(Y^P)+Var(Y^T)} < 1$ $\quad$ $\hat{a} = \bar{C} - \hat{Y} = (1-\hat{b})\hat{Y} \neq 0$ $\quad$ $\Rightarrow\downarrow APC$ as $\uparrow Y$

then in the LR and under the structural model $C^P = Y^P$ and $a = 0, b = 1$, but in the SR we get the Keynesian result $(a > 0, b < 1)$ due to noise variables (temporary shocks). .

In summary, in the SR: $V(Y^T)$ matters (no change in Y^P), but in the LR: $V(Y^P) > V(Y^T)$ and $b^{LR} \approx 1$.

2.1 Framework with Rational Expectations:

Set up: UMP by HH is,

$$v(x_0) = \max_{\{c_t\}_{t=0}^{\infty}} E_0 \sum_{t=0}^{\infty} \beta^t u(c_t, \dots) \quad s.t. \quad \begin{array}{l} c_t \in \Gamma^c(x_t) \\ x_{t+1} \in \Gamma^x(x_t, c_t, \tilde{R}_{t,t+1}, \tilde{y}_{t+1}, \tilde{y}_t) \end{array}$$

here x_t: Assets, $\tilde{y}_t$: income, R: gross return.
Usual assumptions: $u(\cdot)$ concave, Inada conditions, No ponzi game: $\lim_{T\to\infty}\left(\frac{1}{1+r}\right)^T x_T \leq 0$
Discount factor: β, Discount rate: (perc. change of discount factor) $\delta = \frac{\beta^{t-1}-\beta^t}{\beta^t} = \frac{1}{\beta} - 1$

[BE]: $v(x_t) = \max\{u(c_t) + \beta E v(x_{t+1})\}$

$$\begin{aligned} & c_t \in \Gamma(x_t) = [0, x_t] \\ & x_{t+1} = \tilde{R}_{t+1}(x_t - c_t) + \tilde{y}_{t+1} \\ & x_0 = y + 0 \end{aligned}$$

FOC: $u'(c_t) = \beta E_t v'(x_t + 1)\tilde{R}_{t+1}$

Envelope: $v'(x_{t+1}) = u'(c_{t+1})$ $\quad\Rightarrow$

$$\boxed{\text{[EE]: } \mathbf{u'(c_t) = \boldsymbol{\beta} E_t \tilde{R}_{t+1} u'(c_{t+1})}}$$

Perturbation argument for EE: start with $\{c_t^*\}_{t=0}^{\infty}$, apply a small perturbation at t to c_t^* such that $\Delta c_t^* = -\delta$, save it and consume it in $t+1$: $\Delta c_{t+1} = \delta(E_t\tilde{R}_{t+1}) \to$ $\quad \Delta v(x) = -\delta u'(c_t^*) + \beta\delta(E_t\tilde{R}_{t+1})$, take δ small, i.e.,

$\Delta v(x) \approx 0 \quad \Rightarrow \quad u'(c_t) = \beta E_t \tilde{R}_{t+1} u'(c_{t+1})$, also, if the latter equation holds with inequality then shift c_t, c_{t+1} until they are equal.

2.1.1 Empirical tests

Testing the EE: $u'(c_t) = \frac{1}{1+\delta} E_t[u'(c_{t+1}(1+r_t))]$

Simplifying assumptions: Non stochastic r_t and $r_t = \delta \quad \Rightarrow \quad u'(c_t) = E(u'(c_{t+1}))$ and by law of iterated expectations $u'(c_t) = Eu'(c_{t+n}) \quad \forall n \Rightarrow$ MU is a RW.

Also if $u(\cdot)$ is quadratic then $u'(\cdot)$ is linear, $u'(\cdot) = a + 2bc_t \quad \Rightarrow \boxed{c_t = E_t c_{t+1}}$ and c_t is a RW. Using the BC we solve for c_t^*: $c_t = \frac{1}{\sum_{t=0}^{\infty}(1+r)^{-s}} \left[\sum_{t=0}^{\infty}(1+r)^{-s} E_t \tilde{Y}_{t+s} + x_t\right]$

Hall's test: Excess Sensitivity

Given $c_t = E_t c_{t+1} \quad \Rightarrow \quad \Delta c_{t+1} = \varepsilon_{t+1}$ (RE)

Test: $\Delta c_{t+1} = \alpha + \beta' x_t + e_{t+1}$ (Joint test of: Euler Eq, $r = \rho$,quadratic utility, rational exp.)
$H_0: \quad \alpha = \beta = 0$ and $R^2 \approx 0$ or low.

however the result is that Δy_t is significant, i.e., $\beta_y > 0 \quad \Rightarrow \quad E(\Delta c_{t+1} | z_t \in I_t)$ (excess sensitivity: change in consumption is sensitive to exogenous variables.)

Linearizing the EE

- CRRA $u(\cdot)$: $\quad u(c) = \frac{c^{1-\gamma}-1}{1-\gamma}, \qquad u'(c) = c^{-\gamma}$

  replace in EE : $u'(c) = \beta E\left[(1+r_{t+1})c_{t+1}^{-\gamma}\right]$

  $$c_t^{-\gamma} = \frac{1}{1+\rho} E\left[(1+r_{t+1})c_{t+1}^{-\gamma}\right]$$

  $$1 = E\left[\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma}\left(\frac{1+r_{t+1}}{1+\rho}\right)\right]$$

  $$1 = E_t e^{\ln\left[\left(\frac{c_{t+1}}{c_t}\right)^{-\gamma}\left(\frac{1+r_{t+1}}{1+\rho}\right)\right]}$$

  $$\boxed{1 = E_t e^{(r_{t+1} - \rho - \gamma \Delta \ln c_{t+1})}} \quad \text{(since } \ln(c_{t+1}/c_t) = \Delta ln c_{t+1}, \ln(1+x) \approx x)$$

- r_{t+1} known at t

- $\Delta \ln c_{t+1} \sim N$ and $\ln c \sim N$ i.e., consumption is lognormal, its log is normal and then: $Ee^{\ln c} = e^{E \ln c + \frac{1}{2} var(\ln c)}$

  $$1 = e^{\left[E_t r_{t+1} - \rho - \gamma E_t \Delta \ln c_{t+1} + \frac{1}{2}\gamma^2 Var_t \Delta \ln c_{t+1}\right]} \quad \Rightarrow \quad E_t \Delta \ln c_{t+1} = \frac{1}{\gamma}\left(E_t r_{t+1} - \rho\right) + \underbrace{\frac{1}{2}\gamma Var_t \Delta \ln c_{t+1}}_{\text{Precautionary Savings}}$$

  with RE: $\Delta \ln c_{t+1} = \frac{1}{\gamma}\left(E_t r_{t+1} - \rho\right) + \frac{1}{2}\gamma Var_t \Delta \ln c_{t+1} + \varepsilon_{t+1}$

- Assume precautionary savings are constant over time

  $\Delta \ln c_{t+1} = \alpha + \frac{1}{\gamma} r_{t+1} + \varepsilon_{t+1}$

  Conclusion: C depends on r_t, if $r_t = \rho$ or another constant then the consumption would be a RW but this is not generally the case. When r_{t+1} is high then giving up current consumption yields more in the future, then an increase in $\Delta \ln c_{t+1}$ is to be expected.

- Empirical test of linearized EE:

Estimate elasticity of intertemporal substitution:[3]

$EIS \equiv \frac{\partial \Delta \ln c_{t+1}}{\partial r_{t.t+1}} = \frac{1}{\gamma}$ (under CRRA)

Test of Orthogonality restriction $\Omega \perp \varepsilon_{t+1}$ (no exogenous variable matters, e.g. add $\beta \Delta \ln y_{t+1}$)
Result: $\Delta \ln c_{t+1} = \alpha = \frac{1}{\gamma} r_t + \beta \Delta \ln y_{t+1}$ (income matters!)
$\frac{1}{\gamma} \approx 0$ (huge γ) and $\beta > 0$, $\beta \in [0.1, 0.8]$

Why Expected Income predicts consumption?:

- Leisure and consumption are substitutes

- Lifecycle: household supports more people when income is high.

- Precautionary savings not constant overtime → ommited variable bias.

- Alternative preference specifications: Non additive separable utility, need a more general $u(\cdot)$ than CRRA.

- Buffer stock models: (liquidity constraints + impatience) Agents face borrowing constraints and then cannot smooth consumption, consumers are also impatient ($\rho > r$) then there is a buffer stock type of consumption, i.e., agents accumulate small stock of assets to buffer transitory income shocks, after that they just consume y_t leftover.

- People don't optimize (subrationality) but follows rules of thumb.

- **$\mathbf{r_t}$ is stochastic.**

2.2 Link with Asset Returns (CAPM based models and Puzzles)

Consumption CAPM: Multiple assets with stochastic return $\tilde{r}_t^i$

EE holds for each asset i: $E_t \left[\frac{u'(c_{t+1})}{u'(c_t)} \frac{1+\tilde{r}_t^i}{1+\rho} \right] = 1 \quad \forall i$
Stochastic Discount Factor (pricing kernel): $\tilde{M}_t = \frac{u'(c_{t+1})}{u'(c_t)} \frac{1}{1+\rho}$, then:

$$E_t \left[\tilde{M}_t (1 + \tilde{r}_t^i) \right] = 1 \qquad \text{(replace SDF)}$$

$$E_t \tilde{M}_t E_t (1 + \tilde{r}_t^i) + Cov \left(\tilde{M}_t, 1 + \tilde{r}_t^i \right) = 1 \qquad \text{(Rewrite Expectation of a product)}$$

$$E_t (1 + \tilde{r}_t^i) = \frac{1}{E_t \tilde{M}_t} \left[1 - Cov \left(\tilde{M}_t, 1 + \tilde{r}_t^i \right) \right]$$

$$E_t (1 + \tilde{r}_t^i) = \frac{1}{E_t \tilde{M}_t} \left[1 - \beta Cov \left(1 + \tilde{r}_t^i, \left(\frac{c_{t+1}}{c_t} \right)^{-\gamma} \right) \right] \qquad \text{under CRRA, replacing: } \tilde{M}_t = \beta (\tfrac{c_{t+1}}{c_t})^{-\gamma}$$

$$E_t (1 + \tilde{r}_t^i) = \frac{\beta}{E_t \tilde{M}_t} \left[1 + \rho + \gamma Cov_t \left(\tilde{r}_t^i, \Delta \ln c_{t+1} \right) \right] \qquad \text{where } 1/\beta = 1 + \rho$$

Therefore for the agent to hold an asset i (LHS) there should be an increase/compensation in the return the more risk averse and the more correlated the return with the consumption growth (RHS). The intuition behind is that any agent would prefer to diversify their assets by holding assets negatively correlated with his income so that they would be covered in recessions.

Assuming log normality of asset return and consumption growth we can approximate:

$$E \left(1 + \tilde{r}_t^i \right) \approx 1 + \rho + \gamma Cov \left(\tilde{r}_t^i, \Delta \ln c_{t+1} \right) - \frac{\gamma(1+\gamma)}{2} Var(\Delta \ln c_{t+1}) + \gamma E \left(\Delta \ln c_{t+1} \right)$$

---

[3] Generally $EIS_{xy} = -\frac{\partial \ln \left(\frac{x}{y} \right)}{\partial \ln \left(\frac{u_x}{u_y} \right)}$

Then with stochastic returns the return reflects more than just the impatience. Depending on the degree of risk aversion it is influenced by the covariance with consumption and the variance of consumption growth (precautionary savings term).

2.2.1 Equity Premium Puzzle (EPP)

Risk free asset: f; Risky asset (equity): e

Using C-CAPM: $\boxed{E(\tilde{r}_t^e) - \tilde{r}_t^f \propto \gamma Cov(\tilde{r}_t^e, \Delta \ln c_{t+1}) = \gamma \sigma_{ec}}$

This equation is tested/estimated. Empirically LHS ≈ 0.06 whereas the covariance term (σ_{ec}) lies around $0.0003 - 0.0024$ (low covariance), which leads to the EPP: $\gamma = \frac{0.06}{\sigma_{ec}} = [20, 200]$ i.e. a huge risk aversion coefficient (the expected is 2 or 3).

A high γ is a problem since it implies indifference conditions in lotteries that yield an extremely low profit (Mankiw, Zeldes 1991). An explanation for this troublesome result is that the consumption doesn't move much, causing the covariance to be very small.

2.2.2 Risk free rate puzzle

: discount rate becomes negative ($\rho < 0$)

Return observed in equity markets is not consistent with C-smoothing behavior.

Use C-CAPM with the risk free asset: $E\Delta \ln c_{t+1} = \frac{r_t^f - \rho}{\gamma} + \frac{1+\gamma}{2} Var(\Delta \ln c_{t+1})$ (the covariance term is zero)

solve for discount rate: $\rho = r_t^f - \gamma E \Delta \ln c_{t+1} + \frac{(1+\gamma)\gamma}{2} Var \Delta \ln c_{t+1}$

Puzzle: if you take $\gamma = 20, r_t^f = 0.01, \Delta \ln c_{t+1} \approx 0.018, Var(\Delta \ln c_{t+1}) \approx 0.001 \Rightarrow \rho \approx -0.15 < 0$ (**negative DF, i.e., marginal utility in the future is higher.**)

Possible explanations for EPP:

- Calibration done with aggregate data ignores heterogeneous agents and individual dynamics (averaging discards consumption movement).

- Only a small fraction of HH hold equities and they have higher covariance.

- Rare events: small chances of catastrophic events lead to overestimation of average return $E(r^e_{data}) > E(r^e_{\text{market distribution}})$

- Consumption response could be delayed so $Cov(\cdot)$ estimates could be biased downwards.

- Loss aversion: households weight losses twice as heavily than gains.

- Survivorship bias: stock markets have broke down and surviving stocks are not a random sample (same as rare events, implies overestimation of returns or higher premium).

Extra:
Hansen-Jagannathan Bound: the Aim is to set the Stochastic Discount Factor that is consistent with the data (reverse direction than before, i.e., Data → implications on model (SDF)).
Source: Pennachi, Theory of Asset Pricing (2007)

(context) EPP: for reasonable levels of risk aversion, aggregate consumption appears to vary too little to justify the high Sharpe ratio (excess return over volatility) for the market portfolio of stocks.

Aim: in a complete market, create a portfolio that replicates the SDF $\tilde{m}$, i.e., create an asset with return $\tilde{R}_m$ s.t. $Cov(\tilde{R}_m, \tilde{m}_{t+1}) = Var(\tilde{R}_m)$

Depart from the basic Pricing Equation $P_t = E[\tilde{m}_{t+1}X_{t+1}]$ where X_{t+1} denotes future cashflows.

Express it in terms of returns and rearrange (we drop the time subscripts for simplicity):

$$\begin{aligned} 1 &= E[\tilde{m}\tilde{R}] \\ 1 &= E[\tilde{m}]E[\tilde{R}] + Cov(\tilde{m}, \tilde{R}) \\ 1 &= \frac{1}{R_F}\bar{R} + Cov(\tilde{m}, \tilde{R}) \\ \bar{R} - R_F &= -R_F Cov(\tilde{m}, \tilde{R}) \end{aligned}$$

Apply it to the replicated portfolio with return $\tilde{R}_m$:

$$\bar{R}_m - R_F = -R_F\sigma_m^2 \;\Rightarrow\; \underbrace{\frac{\bar{R}_m - R_F}{\sigma_m}}_{\text{Sharpe Ratio of m}} = -R_F\sigma_m$$

For every other asset/portfolio: $\frac{\bar{R}-R_F}{\sigma} = -R_F\rho_{m,R}\sigma_m = \frac{\bar{R}_m - R_F}{\sigma_m}$

Apply absolute value and consider the maximum correlation:

$$\frac{|\bar{R} - R_F|}{\sigma} \leq \underbrace{\frac{|\bar{R}_m - R_F|}{\sigma_m}}_{\text{H-J Upper Bound}}$$

Use the CRRA and the observed values for some parameters:

$$\begin{aligned} \frac{|\bar{R}_m - R_F|}{\sigma_m} = R_F\sigma_m = (e^{\gamma^2\sigma_c^2} - 1)^{\frac{1}{2}} &\geq \overset{\substack{\text{Empirical Mkt.} \\ \text{Sharpe Ratio}}}{\frac{1}{2}} \\ e^{\gamma^2\sigma_c^2} &\geq \frac{1}{4} + 1 \\ \gamma^2\sigma_c^2 &\geq \log(\frac{5}{4}) \approx 0.223 \end{aligned}$$

Therefore for $\sigma_c = 0.04$ we have $\gamma \geq 12$.

Intuition: (i) Given the volatility of the SDF, sets an upper bound on the Sharpe ratio an asset can attain; or (ii) Given an asset's Sharpe ratio and the risk free rate, sets a lower bound on the feasible volatility of the economy's SDF.

Problem: Even after considering this bound, i.e., rationalizing higher values of risk aversion to provide some explanation to the EPP, the risk-free rate puzzle remains ($r < 0$ or $\beta > 1$).

Key issue: most of these puzzling results obey to the fact that by construction the CRRA links the risk aversion (γ) and the IES ($\frac{1}{\gamma}$). This lacks empirical support since it implies that the more risk averse, the lower the degree of substitution for an agent.

Answer to this problem: use *Epstein-Zin* preferences (recursive formulation).

Another one: "Asset Pricing with Garbage", Savov, JoF 2011. Uses garbage data by household as proxy to consumption and obtains new data for consumption that is more volatile and is compatible with $17 < \gamma < 87$ and evades the risk free rate puzzle. This is the current closest approach to solve (empirically) the EPP.

Note: above we used the CRRA so that the following results hold,

$$
\begin{aligned}
m_{t+1} &= \beta \left(\frac{c_{t+1}}{c_t} \right)^{-\gamma} \\
E[m_{t+1}] &= \beta E[e^{-\gamma \tilde{z}}] = \beta e^{-\gamma\mu + \gamma^2\sigma^2/2} \\
m_{t+1}^2 &= \beta^2 \left(\frac{c_{t+1}}{c_t} \right)^{-2\gamma} \\
E[m_{t+1}^2] &= \beta^2 e^{-2\gamma\mu + 4\gamma^2\sigma^2/2} \\
Var(m_{t+1}) &= E[m_{t+1}^2] - E[m_{t+1}]^2 = \beta^2 e^{-2\gamma\mu + \gamma^2\sigma^2} (e^{\gamma^2\sigma^2} - 1) \\
\frac{\sigma_{m_{t+1}}}{E[m_{t+1}]} &= \cancel{\frac{\beta e^{-\gamma\mu + \gamma^2\sigma^2/2}}{\beta e^{-\gamma\mu + \gamma^2\sigma^2/2}}} (e^{\gamma^2\sigma^2} - 1)^{\frac{1}{2}}
\end{aligned}
$$

2.3 Intertemporal Elasticity of Substitution (methodology)

$$ies_t(c_{t+1}, c_t) = -\frac{\frac{d\left(\frac{c_t+1}{c_t}\right)}{\frac{c_{t+1}}{c_t}}}{\left[\frac{d\left(\frac{\frac{\partial u(c)}{\partial c_{t+1}}}{\frac{\partial u(c)}{\partial c_t}}\right)}{\frac{\frac{\partial u(c)}{\partial c_{t+1}}}{\frac{\partial u(c)}{\partial c_t}}}\right]} = -\left[\frac{\frac{d\left(\frac{\frac{\partial u(c)}{\partial c_{t+1}}}{\frac{\partial u(c)}{\partial c_t}}\right)}{d\left(\frac{c_t+1}{c_t}\right)}}{\frac{\frac{\frac{\partial u(c)}{\partial c_{t+1}}}{\frac{\partial u(c)}{\partial c_t}}}{\frac{c_{t+1}}{c_t}}}\right]^{-1}$$

i.e. the IES is the inverse of the percentage change in the marginal rate of substitution between consumption at t and $t+1$ in response to a percentage change in the consumption ratio $\frac{c_{t+1}}{c_t}$

for the CRRA utility function:

$$\frac{\frac{\partial u(c)}{\partial c_{t+1}}}{\frac{\partial u(c)}{\partial c_t}} = MRS(c_{t+1}, c_t) = \beta\left(\frac{c_{t+1}}{c_t}\right)^{-\sigma}$$

then:

$$ies_t(c_{t+1}, c_t) = -\left[\frac{-\sigma\beta\left(\frac{c_{t+1}}{c_t}\right)^{-\sigma-1}}{\frac{\beta\left(\frac{c_{t+1}}{c_t}\right)^{-\sigma}}{\frac{c_{t+1}}{c_t}}}\right] = \frac{1}{\sigma}$$

the intertemporal elasticity of substitution is constant and equal to $\frac{1}{\sigma}$.

The IES measures the curvature of the utility function: If $\sigma = 0$ consumption is adjacent periods are perfect substitutes and $ies = \infty$, if $\sigma \to \infty$ the utility function converges to a Leontieff function, consumption in adjacent periods are perfect complements, $ies = 0$.

Additionally from the FOC of the HH UMP we get:

$$\frac{\frac{\partial u(c)}{\partial c_{t+1}}}{\frac{\partial u(c)}{\partial c_t}} = \frac{p_{t+1}}{p_t} = \frac{1}{1+r_{t+1}}$$

and then we can write the IES as:

$$ies_t(c_{t+1}, c_t) = -\frac{\frac{d\left(\frac{c_t+1}{c_t}\right)}{\frac{c_{t+1}}{c_t}}}{\left[\frac{d\left(\frac{\frac{\partial u(c)}{\partial c_{t+1}}}{\frac{\partial u(c)}{\partial c_t}}\right)}{\frac{\frac{\partial u(c)}{\partial c_{t+1}}}{\frac{\partial u(c)}{\partial c_t}}}\right]} = -\left[\frac{\frac{d\left(\frac{c_{t+1}}{c_t}\right)}{\frac{c_{t+1}}{c_t}}}{\frac{d\left(\frac{1}{r_{t+1}}\right)}{\frac{1}{r_{t+1}}}}\right]$$

i.e. the IES can also be expressed as the percentual change in the consumption growth with respect to the percentual change in the gross real interest rate.

Notice that for the CRRA the EE is:

$$(1 + r_{t+1})\beta\left(\frac{c_{t+1}}{c_t}\right)^{\sigma} = 1$$

take logs:

$$\ln(1 + r_{t+1}) + \ln\beta = \sigma[\ln c_{t+1} - \ln c_t]$$

rearrange:

$$\ln c_{t+1} - \ln c_t = \frac{1}{\sigma}\ln\beta + \frac{1}{\sigma}\ln(1 + r_{t+1})$$

this equation is the basis from IES estimates.

Finally with the CRRA the attitute of the HH towards risk is measured by the risk aversion coefficient σ and the attitude towards consumption smoothing is measures by the IES $\frac{1}{\sigma}$. The fact that these two are determined by the same parameter is an undesirable restriction that generates all the problems mentioned in the consumption exercises above.

3 Labor Frictions: DMP search and matching model

Matching function: $m(u_t, v_t) = k u_t^{\alpha} v_t^{1-\alpha}$ (u: unemployed workers, v: vacancies)

Probabilities:

- of being matched: $x_t = \frac{m(u_t,v_t)}{u_t} = k\frac{u_t^{\alpha} v_t^{1-\alpha}}{u_t} = \frac{v_t^{1-\alpha}}{u_t^{1-\alpha}} = k\theta^{1-\alpha}$
- of filling a vacancy: $y_t = \frac{m(u_t,v_t)}{v_t} = k\frac{u_t^{\alpha} v_t^{1-\alpha}}{v_t} = k\frac{u_t^{\alpha}}{v_t^{\alpha}} = k\theta^{-\alpha}$

then: $\mathbf{x_t = k}\theta^{\mathbf{1}-\alpha}$ and $\mathbf{y_t = k}\theta^{-\alpha}$ where θ : tightness of labour market (vacancies by unemployed).

Break up rate: s

Value functions:

- Firm: $J(p_t, w) = p_t - w + \delta(1-s)E(J(p_{t+1}, w))$
- Worker: $W(p_t, w) = w + \delta s E[U(p_{t+1})] + \delta(1-s)E[W(p_{t+1}, w)]$
- Unemployed: $U(p_t) = z + \delta\left[x_t E(W(p_{t+1}, w)) + (1-x_t)E(U(p_{t+1}))\right]$

from these values funtions we can determine:

$$\frac{\partial J(p_t,w)}{\partial w} = -1 + \delta(1-s)E\left[\frac{\partial J(p_{t+1},w)}{\partial w}\right] \rightarrow \frac{\partial J(p_t)}{\partial w} = \frac{-1}{1-\delta(1-s)} < 0$$
$$\frac{\partial W(p_t,w)}{\partial w} = 1 + \delta(1-s)e\left[\frac{\partial w(p_{t+1},w)}{\partial w}\right] \rightarrow \frac{\partial W(p_t,w)}{\partial w} = \frac{1}{1-\delta(1-s)} > 0$$

The surplus of a match is given by: $S(p_t, w) = w(p_t, w) - U(p_t) + J(p_t, w)$

Let the surplus at the optimal wage be: $V(p_t) \equiv S(p_t, w^*)$

Assume the optimal wage is chosen by Nash bargaining: $w = \underset{\hat{w}}{\arg\max}\,(W(p_t, \hat{w}) - U(p_t))^{\beta} J(p_t, \hat{w})$

FOC:

$$[w] : \beta\frac{\partial W(p_t,w)}{\partial w}(W(p_t,w) - U(p_t,w))^{\beta-1}J(p_t,w)^{1-\beta} = -\frac{\partial J(p_t,w)}{\partial w}(1-\beta)(W(p_t,w) - U(p_t,w))^{\beta-1}J(p_t,w)^{1-\beta}$$
$$\beta\frac{1}{1-\delta(1-s)}(W(p_t,w) - U(p_t,w))^{\beta-1}J(p_t,w)^{1-\beta} = \frac{1}{1-\delta(1-s)}(1-\beta)(W(p_t,w) - U(p_t,w))^{\beta-1}J(p_t,w)^{1-\beta}$$
$$(1-\beta)\left(\frac{W-U}{J}\right)^{\beta} = \beta\left(\frac{J}{W-U}\right)^{1-\beta} \quad \Rightarrow \quad \boxed{J(p_t,w) = \frac{1-\beta}{\beta}(W(p_t,w) - U(p_t))}$$

Subs. in $S(p_t, w)$: $S(p_t, w) = \frac{W(p_t,w)-U(p_t)}{\beta}$ (+)

Also:

$$S(p_t,w) = w - z + p_t - w + \delta(1-s)\left(E\left[W(p_{t+1},w)\right] + E\left[J(p_{t+1},w)\right] - E\left[U(p_{t+1})\right]\right) - \delta k\theta_t^{1-\alpha}\left(E\left[W(p_{t+1,w})\right] - E\left[U(p_{t+1,w})\right]\right)$$

using (+): $S(p_t, w) = p_t - z + \delta(1-s)E\left[V(p_{t+1})\right] - \delta k\theta_t^{1-\alpha}\beta E\left[V(p_{t+1})\right]$

$$S(p_t, w) = p_t - z - \delta(\beta k\theta_t^{1-\alpha} - 1 + s)E\left[V(p_{t+1})\right] \qquad (\star)$$

Free entry condition:

Exp. benefits of filling a vacancy = Cost of filling a vacancy

$$y_t J(p_t, w) + (1-y_t)0 = C$$
$$y_t(1-\beta)V(p_t) = C \text{ (replace J in terms of V)}$$
$$\frac{k}{\theta_t^{\alpha}}(1-\beta)V(p_t) = C \text{ (subs. } y_t = \theta^{1-\alpha})$$
$$V(p_t) = \frac{\theta_t^{\alpha}C}{k(1-\beta)}\forall t \quad \rightarrow \quad V(p_{t+1}) = \frac{\theta_{t+1}^{\alpha}C}{k(1-\beta)}$$

subs. in value function for surplus from $(\star)$: $\frac{C\theta_t^{\alpha}}{k(1-\beta)} = p_t - z - \frac{\delta(\beta k\theta_t^{1-\alpha}-1+s)}{k(1-\beta)}E\left[C\theta_{t+1}^{\alpha}\right]$

In SS: $e\left[\theta_{t+1}^{\alpha}\right] = \theta^{\alpha_t} \quad \Rightarrow \quad \frac{C\theta_t^{\alpha}}{k(1-\beta)} = p_t - z - \frac{C\delta\beta k\theta_t - C\delta(1-s)\theta_t^{\alpha}}{k(1-\beta)} \qquad (\clubsuit)$

find $\frac{\partial\theta_t}{\partial(p_t-z)}$: $\quad \frac{C\alpha\theta_t^{\alpha-1}}{k(1-\beta)}\frac{\partial\theta_t}{\partial p_t - z} = 1 - \left(\frac{C\delta\beta k - C\delta(1-s)\alpha\theta_t^{\alpha-1}}{k(1-\beta)}\right)\frac{\partial\theta_t}{\partial p_t - z}$

at $\theta = 1$: $\quad \frac{C\alpha}{k(1-\beta)}\frac{\partial\theta_t}{\partial p_t - z} = 1 - \left(\frac{C\delta\beta k - C\delta(1-s)\alpha}{k(1-\beta)}\right)\frac{\partial\theta_t}{\partial p_t - z}$

then, $\quad \frac{\partial\theta_t}{\partial p_t - z} = \frac{k(1-\beta)}{C(\alpha+\delta(\beta K-(1-s)\alpha))}$

Now we can find the elasticity of market tightness to produvity $(p_t - z)$:

$$\varepsilon_{\theta_t,p_t-z|\theta=1} = (p_t - z)\frac{\partial\theta_t}{\partial p_t - z} = \frac{C+C\delta\beta k-C\delta(1-s)}{K(1-\beta)}\frac{k(1-\beta)}{C(\alpha+\delta(\beta k-(1-s)\alpha))}$$

$$\varepsilon_{\theta_t,p_t-z|\theta=1} = \frac{1+\delta(\beta k-(1-s))}{\alpha+\delta(\beta k-(1-s)\alpha)}$$

Only if $\theta = 1$ the elasticity to vacancies-unemployed ratio to productivity would be 1 which means that one would expect the vacancies to respond strongly to changes in productivity, potentionally explaining changes in the business cycle. However, when the rest of calibrated parameters are taken into account the elasticity is roughly above 1 ($\varepsilon_{\theta,p-z} \approx 1.03$) which means that vacancies don't really react strongly after changes in productivity have taken place.

4 Dynamic Programming

$(SP \longleftrightarrow BE)$:

$$v(x_0) = \max_{x_{+1}\in\Gamma(x)} \sum \delta^t F(x, x_{+1}) \quad \longleftrightarrow \quad v(x) = \max_{x_{+1}\in\Gamma(x)} F(x_0, x_1) + \delta v(x_{+1})$$

"$\rightarrow$"

$$\begin{aligned} v(x_0) &= \max_{x_{+1}\in\Gamma(x)} \textstyle\sum_{t=0}^{\infty} \delta^t F(x_t, x_{t+1}) \\ &= \max_{x_{+1}\in\Gamma(x)} \{F(x_0, x_1) + \textstyle\sum_{t=1}^{\infty} \delta^t F(x_t, x_{t+1})\} \\ &= \max_{x_{+1}\in\Gamma(x)} \{F(x_0, x_1) + \delta \textstyle\sum_{t=1}^{\infty} \delta^{t-1} F(x_t, x_{t+1})\} \\ &= \max_{x_{+1}\in\Gamma(x)} \{F(x_0, x_1) + \delta \max_{x_{+2}\in\Gamma(x_{+1})} \textstyle\sum_{t=0}^{\infty} \delta^t F(x_{t+1}, x_{t+2})\} \\ &= \max_{x_{+1}\in\Gamma(x)} \{F(x_0, x_1) + \delta v(x_1)\} \end{aligned}$$

"$\leftarrow$"

$$\begin{aligned} v(x_0) &= \max_{x_1\in\Gamma(x_0)} \{F(x_0, x_1) + \delta v(x_1)\} \\ &= \max_{x_1\in\Gamma(x_0)} \{F(x_0, x_1) + \delta\left[F(x_1, x_2) + \delta v(x_2)\right]\} \\ &\vdots \\ &= \max_{x_1\in\Gamma(x_0)} \{F(x_0, x_1) + \cdots + \delta^{n-1} F(x_{n-1}, x_n) + \delta^n v(x_n)\} \\ &= \max_{x_1\in\Gamma(x_0)} \{\textstyle\sum_{t=0}^{\infty} \delta^t F(x_t, x_{t+1})\} \end{aligned}$$

where "$\leftarrow$" holds if $\lim_{n\to\infty} \delta^n v(x_n) = 0$

Example: Optimal growth

In SP notation:

$$v(k_0) = \max_{\{k_{t+1}\}_{t+0}^{\infty}} \textstyle\sum_{t=0}^{\infty} \delta^t \ln(k_t^{\delta} - k_{t+1})$$

s.t. $k_{+1} \in \Gamma(k) \equiv [0, k^{\alpha}]$ and k_0 given.

In BE notation:

$$v(k) = \max_{k_{+1}\in\Gamma(k)} \{\ln(k^{\alpha} - k_{+1})\}$$

$\forall k$

4.1 Methods of solution:

Guess a form and check, iterative solution, numerical solution.

4.1.1 Guess and check

$$V(x_t) = \sup_{x_{t+1}\in\Gamma(x_t)} \{F(x_t, x_{t+1}) + \beta V(x_{t+1})\}$$

FOC:

$$[x_{t+1}] : \qquad \frac{\partial F(x_t,x_{t+1})}{\partial x_{t+1}} + \beta V'(x_{t+1}) = 0$$

Envelope condition: (derivative of whole expression —on both sides— wrt x_t. Implies using chain rule as x_t also affects $x_{t+1} = \Gamma(x_t)$)

$$V'(x_t) = \frac{\partial F(x_t,x_{t+1})}{\partial x_t} + \frac{\partial F(x_t,x_{t+1})}{\partial x_{t+1}}\frac{dx_{t+1}}{dx_t} + \beta V'(x_{t+1})\frac{dx_{t+1}}{dx_t}$$

$$= \frac{\partial F(x_t,x_{t+1})}{\partial x_t} + \underbrace{\left[\frac{\partial F(x_t,x_{t+1})}{\partial x_{t+1}} + \beta V'(x_{t+1})\right]}_{\text{FOC w.r.t. } x_{t+1}=0}\frac{dx_{t+1}}{dx_t} \quad \Rightarrow \quad \boxed{V'(x_t) = \frac{\partial F(x_t,x_{t+1})}{\partial x_t}}$$

forward: $V'(x_{t+1}) = \frac{\partial F(x_{t+1},x_{t+2})}{\partial x_t}$, subs. in FOC: $\boxed{\frac{\partial F(x_t,x_{t+1})}{\partial x_{t+1}} = -\beta\frac{\partial F(x_{t+1},x_{t+2})}{\partial x_{t+1}}}$ (Euler Equation)

4.1.2 Iteration of the Bellman equation

Bellman functional operator: T

$(Tw)(x) = \sup_{x_{+1}\in\Gamma(x)} \{F(x, x_{+1}) + \beta w(x_{+1})\}$ $\forall x$, T maps a funtion $w(\cdot)$ to $Tw(\cdot)$ (functional operator). Let v be a solution and a fixed point, i.e., if $w = v$ then $Tw = w$:

$$(Tv)(x) = \sup_{x_{+1}\in\Gamma(x)} \{F(x, x_{t+1}) + \beta \underbrace{(\sup\{F(x_{+1}, x_{+2}) + \beta v(x_{+2})\})}_{v(x_{+1})}\}$$

$$(Tv)(x) = \sup_{x_{+1}\in\Gamma(x)} \{F(x, x_{t+1}) + \beta v(x_{+1})\} = v(x)$$

Solution by iteration

Pick some v_0 and iterate $T^n v_0$ ultil convergence:

$$\begin{aligned}(Tw)(x) &= \sup_{x_{+1}\in\Gamma(x)} \{F(x, x_{+1}) + \beta w(x_{+1})\}\\ (T(Tw))(x) &= \sup_{x_{+1}\in\Gamma(x)} \{F(x, x_{+1}) + \beta(Tw)(x_{+1})\}\\ &\vdots\\ (T(T^n w))(x) &= \sup_{x_{+1}\in\Gamma(x)} \{F(x, x_{+1}) + \beta(T^n w)(x_{+1})\}\end{aligned}$$

where $\lim_{n\to\infty} T^n v_0 = v$ and $T^n v_0$ is a Cauchy sequence.

Convergence is guaranteed by the Contraction Mapping Theorem (CMT).

Contraction Mapping Theorem: Let (S,d) be a metric space. The function $T : S \to S$ is a contraction mapping if for some $\beta \in (0, 1)$, $d(Tf, Tg) \leq \beta d(f, g)$ for any $f, g \in S$.

Theorem: If (S,d) is a complete metric space and $T : S \to S$ is a **contraction mapping** then,

i. T has a unique fixed point $v \in S$

ii. $\forall\ v_0 \in S \quad \lim_{n\to\infty} T^n v_0 = v$

iii. $T^n v_0$ has an exponential convergence rate at least as great as $-\ln(\beta)$

the functional operator is a contracting mapping if it satisfies the Blackwell Sufficiency Conditions. The iteration method may not work otherwise.

Blackwell Sufficiency Conditions: Let $X \in \mathbb{R}^L$ and let $C(X)$ be a space of bounded functions $f : X \to \mathbb{R}$ with the sup-metric. Let $T : C(X) \to C(X)$ be an operator satisfying:

Monotonicity: if $f, g \in C(X)$ and $f(x) < g(x) \quad \forall x \in X$ then $(Tf)(x) < (Tg)(x) \quad \forall x \in X$
Discounting: there exists some $\delta \in (0,1)$ such that $(T(f+a))(x) \leq (Tf)(x) + \delta a \quad \forall f \in C(X), a \geq 0, x \in X$

Then T is contraction mapping with modulus δ.

Proof: For any $f, g \in C(X)$, $f(x) \geq g(x) + d(f,g) \forall x$, relaxing notation,

$f \leq g + d(f,g)$, monotonicity and discounting imply:

$Tf \leq T(g + d(f,g)) \leq Tg + \delta d(f,g)$

$Tg \leq T(f + d(f,g)) \leq Tf + \delta d(f,g)$

Combining the last two lines,

$Tf - Tg \leq \delta d(f,g)$

$Tg - Tf \leq \delta d(f,g)$

$|(Tg)(x) - (Tf)(x)| \leq \delta d(f,g) \forall x$

$\sup_x |(Tg)(x) - (Tf)(x)| \leq \delta d(f,g) \quad \Rightarrow \quad d(Tf, Tg) \leq \delta d(f,g)$

4.1.3 Example: NGM

Here we apply the solution methods to the benchmark NGM model.[4]

$U(c) = \ln c$

$F(k,n) = \kappa^\alpha n^{1-\alpha}$

$\delta = 1 \ \Rightarrow f(k) = F(k) + (1-\delta)k_{old} = k^\alpha$

Then:

$$v(k) = \max_{0 \leq k' \leq k^\alpha} \{\ln(k^\alpha - k') + \beta v(k')\}$$

Guess and Verify (method of undetermined coefficients)

1. guess a functional form: $v(k) = A + B \ln k$
2. Solve maximization problem after substituting the guess:

$$\max_{k'} \ln(k^\alpha - k') + \beta[A + B \ln k']$$

   FOC:

   $[k'] : \qquad \frac{1}{\kappa^\alpha - k'} = \frac{\beta B}{k'} \qquad \Rightarrow k' = \frac{\beta B k^\alpha}{1+\beta B}$

3. evaluate the RHS in the optimal solution:

$$RHS = \ln\left(\frac{k^\alpha}{1+\beta B}\right) + \beta A + \beta B \ln\left(\frac{\beta B k^\alpha}{1+\beta B}\right)$$
$$= \alpha \ln(k) - \ln(1+\beta B) + \beta A + \beta B \ln\left(\frac{\beta B}{1+\beta B}\right) + \alpha\beta B \ln(k)$$

---

[4] source: Krueger notes, Chapter 3.

4. Equal LHS and group constant terms:

$$A + B\ln(k) = \underbrace{-\ln(1+\beta B) + \beta A + \beta B \ln\left(\frac{\beta B}{1+\beta B}\right)}_{A} + \underbrace{[\alpha + \alpha\beta B]}_{B}\ln(k)$$

Then: $B = \frac{\alpha}{1-\alpha\beta}$

$A = -\ln(1+\beta B) + \beta A + \beta B \ln\left(\frac{\beta B}{1+\beta B}\right) \xrightarrow{Subs.\ B=\frac{\alpha}{1-\alpha\beta}} A = \frac{1}{1-\beta}\left[\ln(1-\alpha\beta) + \frac{\alpha\beta\ln(\alpha\beta)}{1-\alpha\beta}\right]$

Replace A and B in the policy function:

$$g(k) = k' = \frac{\beta B k^{\alpha}}{1+\beta B} = \frac{\frac{\beta\alpha k^{\alpha}}{1-\alpha\beta}}{1+\frac{\beta\alpha}{1-\beta\alpha}} = \beta\alpha k^{\alpha} \quad \Rightarrow \quad \mathbf{g}(\mathbf{k}) = \alpha\beta\mathbf{k}^{\alpha}$$

then, in this example, the optimal policy consists on saving a fraction $\alpha\beta$ of the output as capital stock and consuming a fraction $1-\alpha\beta$ in each period. The rule doesn't depend on the level of the state variable (capital) but this may change in other setups.

5. construct sequence $\{k_{t+1}\}$ with the policy function:

$$\begin{aligned} k_1 &= g(k_0) = \alpha\beta k_0^{\alpha} \\ k_2 &= g(k_1) = \alpha\beta k_1^{\alpha} = (\alpha\beta)^{1+\alpha} k_0^{\alpha^2} \\ &\vdots \\ k_t &= (\alpha\beta)^{\sum_{j=0}^{t-1}\alpha^j} k_0^{\alpha^t} \end{aligned}$$

with $\lim_{t\to\infty} k_t = \alpha\beta^{\frac{1}{1-\alpha}} = k^*$, k^* is also a fixed point, i.e., the solution of $g(k) = k$.

Value function iteration (analytical approach)

1. guess an arbitrary function $v_0(k)$, e.g., $v_0(k) = 0$
2. solve the optimization problem:

$$\begin{aligned} v_1(k) &= \max_{0\le k'\le k^{\alpha}} \{\ln(k^{\alpha} - k') + \beta \cancelto{0}{v_0(k')}\} \\ &= \max_{0\le k'\le k^{\alpha}} \ln(k^{\alpha} - k') \end{aligned}$$

the solution is given by $k' = 0$. Substitute it in the objective function:

$$v_1(k) = \ln(k^{\alpha} - 0) + \beta v_0(0) = \alpha\ln k$$

we can use this functional form for $v_1(\cdot)$ in the next iteration involving $v_2(\cdot)$.

3. iterate and continue solving:

$$\begin{aligned} v_2(k) &= \max_{0\le k'\le k^{\alpha}} \{\ln(k^{\alpha} - k') + \beta v_1(k')\} \\ &= \max_{0\le k'\le k^{\alpha}} \{\ln(k^{\alpha} - k') + \beta\alpha\ln k'\} \\ &\vdots \\ v_{n+1} &= \max_{0\le k'\le k^{\alpha}} \{\ln(k^{\alpha} - k') + \beta v_n(k')\} \end{aligned}$$

this process is done until convergence, i.e., until the sequences $\{v_n\}_{n=0}^{\infty} \to v^*$ and $\{g_n\}_{n=0}^{\infty} \to g^*$.

Value function iteration (numerical approach)

Suppose that k and k' take values if $\mathcal{K} = \{0.04, 0.08, 0.12, 0.16, 0.2\}$ only. With that the value function will consist of five numbers $(v_n(0.04), v_n(0.08), v_n(0.12), v_n(0.16), v_n(0.2))$. Let $\alpha = 0.3,\ \beta = 0.6$

1. make an initial guess $v_0(k) = 0\ \forall k \in \mathcal{K}$
2. Solve $v_1(k) = \max_{k' \in \mathcal{K}}\{\ln(k^{0.3} - k') + 0.6 \cdot 0\}$. Out of the possible k, the solution is given by $k' = 0.04$. (notice $k' = 0$ is not allowed now)
3. plug $k' = 0.04$ and obtain $v_1(\cdot)$ for the next iteration:

$$v_1(k) = \ln(k^{0.3} - 0.04)$$

The numerical values for $v_1(k)$ are (remember at this point we are not interested in the functional form as much as the numerical values):

$$v_1(0.04) = \ln(0.04^{0.3} - 0.04) = -1.077$$
$$v_1(0.08) = \ln(0.08^{0.3} - 0.04) = -0.847$$
$$v_1(0.12) = \ln(0.12^{0.3} - 0.04) = -0.715$$
$$v_1(0.16) = \ln(0.16^{0.3} - 0.04) = -0.622$$
$$v_1(0.2) = \ln(0.2^{0.3} - 0.04) = -0.55$$

4. continue with more steps: $v_2(k) = \max_{k' \in \mathcal{K}, k' \in [0, k^{0.3}]}\{\ln(k^{0.3} - k') + 0.6 v_1(k')\}$. For each $k \in \mathcal{K}$ get $v_2(k)$, use $v_1(k')$ from the last step. Notice that getting $v_2(k)$ for a given k implies testing each k' and picking the maximum. e.g., for $k = 0.04$, $v_2(0.04) = -1.723$ if $k' = 0.04$, $v_2(0.04) = -1.710$ if $k' = 0.08$ and so on:

for $k = 0.04$:

$$v_2(0.04) = \ln(0.04^{0.3} - 0.04) + 0.6 \cdot \overset{v_1(0.04)}{(-1.077)} = -1.723, \text{ for } k' = 0.04$$
$$v_2(0.04) = \ln(0.04^{0.3} - 0.08) + 0.6 \cdot \overset{v_1(0.08)}{(-0.847)} = -1.710, \text{ for } k' = 0.08$$
$$v_2(0.04) = \ln(0.04^{0.3} - 0.12) + 0.6 \cdot (-0.715) = -1.773, \text{ for } k' = 0.12$$
$$v_2(0.04) = \ln(0.04^{0.3} - 0.16) + 0.6 \cdot (-0.622) = -1.884, \text{ for } k' = 0.16$$
$$v_2(0.04) = \ln(0.04^{0.3} - 0.2) + 0.6 \cdot (-0.55) = -2.041, \text{ for } k' = 0.2$$

i.e., for $k = 0.04$ the optimal choice is $k'(0.04) = g_2(0.04) = 0.08$ which yields a value of $v_2(0.04) = -1.710$.

This process should be carried for every k (and each k' within) yielding:

k \ k'	0.04	0.08	0.12	0.16	0.2
0.04	-1.723	**-1.710**	-1.773	-1.884	-2.041
0.08	-1.493	**-1.453**	-1.482	-1.548	-1.644
0.12	-1.361	**-1.308**	-1.322	-1.369	-1.441
0.16	-1.268	**-1.207**	-1.212	-1.247	-1.305
0.2	-1.196	-1.130	**-1.128**	-1.156	-1.205

→

k	$v_2(k)$	$g_2(k)$
0.04	-1.710	0.08
0.08	-1.453	0.08
0.12	-1.308	0.08
0.16	-1.207	0.08
0.2	-1.128	0.12

5. this process is continued until convergence, i.e., about n times until $v_{n-1} \approx v_n\ \forall k \in \mathcal{K}$.

We can try to generalize what is done in this example in the summary below:

Algorithm 1 Value function interation (numerical approach for NGM model)

```
Initialization: Guess $v_0(k)$, evaluate all $k$ (these values are used in the next iteration.)
Iteration 1:
    For each $k$, determine optimal policy (decision) $k'$. In this step, replace $v_0(k')$ on the RHS of function to maximize from values obtained at the end of past iteration.
    Replace (values for $k'$ for each $k$) and get $v_1(k)$, evaluate for all $k$ (these values are used in next iteration)
    $\Rightarrow$ you got the value function evaluated in the grid for $k$ and the optimal ($k'$) decision: $v_1(k)$, $g_1(k) = k'$
Iteration 2:
    For each $k$, determine optimal policy (decision) $k'$. Make sure to use the appropriate $v_1(k')$ obtained at the end of previous iteration for the continuation values in the last part of the RHS.
    (This determination implies evaluating, for every $k$ in the grid, the function for every $k'$ in the grid. The same was done in previous iterations before implicitly, but if the initial guess is simple, the large quantity of evaluations implied becomes evident only until now.)
    Replace and get $v_2(k)$, evaluate for all $k$ (these values are used in next iteration —as $v_2(k')$ in RHS.)
    After the last step, you obtained the optimal value function and optimal policies (decisions): $v_2(k)$, $g_2(k)$.
repeat
    Continue iterations
until convergence: $v_{n-1} \approx v_n$, $g_{n-1} \approx g_n$ for all $k$ in the grid.
```

4.2 Discrete choice DP: Optimal stopping

The choice might be discrete, e.g., determine the moment to sign a contract, buy/sell a lottery or any one-time decision. A typical example is given by:

An agent draws an offer, x, from an uniform distribution in the unit interval. The agent can either accept the offer and get x or reject and draw a period later (i.e. get the expected continuation value of the game). Rejections are costly because the agent discounts the future exponentially.

If $v(x) = \max\limits_{\text{Accept,Reject}} \{x, \delta E v(x_{+1})\}$ then $\exists x^*$ s.t. $x^* = \delta E[v(x_{+1})]$ the stationary threshold implies there exists a constant $\underline{v}$ s.t.

the goal is to find $g(x)$ that maps x to A or R depending of $v(x)$ optimally. The strategy followed is: 1. Show continuity of $v(x)$, 2. Conjecture a threshold strategy, 3. Use the indifference threshold to choose either A or R.

Conjecture threshold rule x^*: There is some x^* s.t.

$$g(x) = \begin{cases} Accept & \text{if } x \geq x^* \\ Reject & \text{if } x < x^* \end{cases}$$

$g(\cdot)$ would then generate the value function as:

$$v(x) = \begin{cases} x & \text{if } x \geq x^* \\ x^* & \text{if } x < x^* \end{cases}$$

in other words, to find x^* we use the indifference condition (at x^* it holds that $x^* = \delta E[v(x_{+1})]$), i.e., when $x = x^*$ either R or A. Assuming x is uniform in [0,1]:

$$v(x^*) = x^* = \delta E[v(x_{+1})] = \delta \left[\int_{x=0}^{x=x^*} x^* f(x) dx + \int_{x=x^*}^{1} x f(x) dx \right]$$
then: $x^* = \delta x^{*2} + \delta \frac{1}{2}(1 - x^{*2}) = \frac{\delta}{2}(1 + x^{*2}) \quad \Rightarrow \quad x^* = \frac{1-\sqrt{1-\delta^2}}{\delta}$

Notice that as $\delta \to 0$ it's better to accept (A), whereas if $\delta \to 1$ it's better to Reject (R).

Solution by iteration

Start at $v_0(x) = 0$, the Bellman operator is $(Bw)(x) = \max\limits_{A,R}\{x, \delta E[w(x')]\}$

First iteration: $v_1(x) = (Bv_0)(x)$
$$= \max\{x, \delta E v_0(x')\}$$
$$= \max\{x, 0\} = x$$

Since the threshold (x_1) is 0 (continuation value is always 0), $v_1(x) = x$, i.e., A all offers greater or equal to 0.

Second iteration: $v_2(x) = (B^2 v_0)(x)$
$$= B(Bv_0)(x')$$
$$= \max\{x, \delta E B v_0(x')\}$$
$$= \max\{x, \delta E v_1(x')\}$$
$$= \max\{x, \delta E x'\} \qquad \Rightarrow \quad x_2 = \delta E x' = \frac{\delta}{2}$$

In general: $v_n(x) = (B^n v_0)(x)$
$$= B(B^{n-1} v_0)(x)$$
$$= \max\{x, \delta E B^{n-1} v_0(x')\} \qquad \Rightarrow \quad x_n = \delta E B^{n-1} v_0(x')$$

then $v_n(x) = (B^n v_0)(x) = \begin{cases} x_n & \text{if } x \leq x_n \\ x & \text{if } x > x_n \end{cases}$

$x_n = \delta E B^{n-1} v_0(x') \qquad$ with $B^{n-1} v_0(x) = \begin{cases} x_n & \text{if } x \leq x_n \\ x & \text{if } x > x_n \end{cases}$
$$= \delta \left[\int_{x=0}^{x=x_{n-1}} x_{n-1} f(x) dx + \int_{x=x_{n-1}}^{x=1} x f(x) dx \right]$$
$$= \frac{\delta}{2}(1 + \delta^2)$$

if there is convergente $x_n = x_{n-1}$ as $n \to \infty$, then $\lim_{n\to\infty} x_n = \frac{1-\sqrt{1-\delta^2}}{\delta}$ as before in the guessed solution.

5 Basic DSGE Model

- RBC scheme + New Keynesian insights (frictions).

- Business cycles is caused by real shocks but nominal rigidity leads to inefficient outcomes, hence, there is a role for policy.

- Seminal contributions: Clarida, Gertler (1999), Woodford (2003), Gali (2007), Obstfeld and Rogoff (2006).

- Optimization behavior for consumers (HH), producers (Firms) and policy makers.

- Solution methods: Tipically get FOC + log linearization around SS and solve the model. Other solution methods are Blanchard and Kahn, Ulhig, Klein, Sims and Dynare.

Example: Basic NK model

$x_t = E_t x_{t+1} - \phi[i_t - E_t \pi_{t+1}] + \varepsilon_{x,t}$ [Dynamic IS (AD)]

$\pi_t = \beta E_t \pi_{t+1} + \kappa x_t + \varepsilon_{\pi,t}$ [NKPC (AS)]

$i_t = \varphi_\pi \pi_t + \varphi_x x_t + \varepsilon_{i,t}$ [Monetary Policy Rule]

5.1 Dynamic IS

$\max_{C_t, N_t} E_0 \sum_{t=0}^{\infty} \beta^t \left[\frac{C_t^{1-\sigma}}{1-\sigma} - \frac{N_t^{1+\theta}}{1+\theta} \right]$ (Alternative: add $\frac{M}{P}$ (MIU)).

s.t. $P_t C_t + \frac{B_t}{1+i_t} = w_t N_t + B_{t-1}$ B are units of bonds purchased at price $Q_t = \frac{1}{1+i_t}$

CES preferences:

$C_t = \left[\int_0^1 C_t(z)^{\frac{\varepsilon-1}{\varepsilon}} dz\right]^{\frac{\varepsilon}{\varepsilon-1}}$

$P_t = \left[\int_0^1 P_t(z)^{1-\varepsilon} dz\right]^{\frac{1}{1-\varepsilon}}$

then demand for good i: $c_t^i = \left[\frac{P_t^i}{P_t}\right]^{-\varepsilon} C_t$

Set up BE: Take FOC and re-arrange to get EE (intertemporal trade-off) and intratemporal trade-off:

$$[C_t\ ]: \qquad E_t\left\{\beta(1+i_t)\left[\frac{C_{t+1}}{C_t}\right]^{-\sigma}\frac{P_t}{P_{t+1}}\right\} = 1$$

$$[\text{Intra }]\text{jocagraca}: \qquad \frac{-U_{N,t}}{U_{C,t}} = \frac{w_t}{P_t}$$

Log-linearize around SS: c^{ss}

$c_t = E_t c_{t+1} - \frac{1}{\sigma}\left[i_t - E_t\pi_{t+1} - \rho\right] \qquad (\star)$

where c_t is a log deviation of c from c^{ss}.

Market-Clearing:

$c_t^i = y_t^i$

Output gap definition:

$x_t = y_t - \bar{y}_t$ (Natural output defined as flexible price level of output, i.e., no nom rigidities).

subs. and $(\star)$ becomes:

$x_t = E_t x_{t+1} - \frac{1}{\sigma}[i_t - E_t\pi_{t+1}] + \varepsilon_{x,t}$ [Dynamic IS (AD)]

Notice: this equation is now denoted in terms of deviations. To obtain it we needed to subtract the same equation in the steady state, cancelling out the constants like ρ.

5.2 NK Phillips Curve (AS)

- $1-\theta$: probability of price adjustment each period. fraction θ keeps fixed prices.
- Expected time between changes: $\frac{1}{1-\theta}$
- Aggregate prices: $P_t = \left[\theta P_{t-1}^{1-\varepsilon} + (1-\theta)P_t^{*1-\varepsilon}\right]^{\frac{1}{1-\varepsilon}} \qquad (+)$
- Prod. function: $Y_t^i = N_t^i$

Optimal price without nominal rigidity:

$\max\ P_t^i Y_t^i - S_t^n Y_t^i \quad \text{s.t. } Y_t^i = \left[\frac{P_t^i}{P_t}\right]^{-\varepsilon} Y_t$

where S_t^n is the marginal cost of the n good and the BC is defined using the definition of $C^i(Y_t = C_t^i, Y_t = C_t)$ as before.

FOC: after subs. Y_t^i,

$$[P_t^i]: \qquad P_t^{i*} = \overbrace{\frac{\epsilon}{\epsilon-1}}^{\text{mark-up}} S_t^n$$

Sticky prices:

$$\max_{P_t^*} E_t \sum_{t=0}^{\infty}(\beta\theta)^k Q_{t.t+k}\left\{\frac{P_t^*}{P_t}Y_{t+k}(P_t^*) - \frac{S_{t+k}^n}{P_{t+k}}Y_{t+k}(P_t^*)\right\} \quad \text{s.t. } Y_{t+k}(P_t^*) = \left(\frac{P_t^*}{P_{t+k}}\right)^{-\varepsilon} Y_{t+k}$$

here: $Q_{t.t+k} \equiv \frac{U_c(C_{t+k})}{U_c(c_t)} = \left[\frac{C_{t+k}}{C_t}\right]^{-\sigma}$

i.e. $\beta^k Q_{t.t+k} \equiv$ Stochastic Discount Factor between t and $t+k$.

replace $Y_{t+k}(P_t^*)$, (here if Y is not a function of optimal price it refers to the aggregate price)

$$\max_{P_t^*} \; E_t \sum_{t=0}^{\infty}(\beta\theta)^k Q_{t.t+k} Y_{t+k}\left\{\frac{P_t^*}{P_{t+k}}\left[\frac{P_t^*}{P_{t+k}}\right]^{-\varepsilon} - \frac{S_{t+k}^n}{P_{t+k}}\left[\frac{P_t^*}{P_{t+k}}\right]^{-\varepsilon}\right\}$$

FOC:

$[P_t^*]$: $$P_t^* E_t \sum_{k=0}^{\infty}(\beta\theta)^k Q_{t.t+k} Y_{t+k} P_{t+k}^{\varepsilon-1} = \frac{\varepsilon}{\varepsilon-1} E_t \sum_{k=0}^{\infty}(\beta\theta)^k Q_{t.t+k} Y_{t+k} S_{t+k}^n P_{t+k}^{\epsilon-1}$$

log-linearization:

$$p_t^* = (1-\beta\theta)E_t \sum_{k=0}^{\infty}(\beta\theta)^k \overbrace{[S_{t+k}+P_{t+k}]}^{\text{real marginal cost}}$$

Rearrange:

$$p_t^* = (1-\beta\theta)[s_t+p_t] + \beta\theta \; \overbrace{E_t p_{t+1}^*}^{E_t \sum_{t=1}^{\infty}[\dots]} \qquad (X)$$

On the other hand, log linerarize (+):

$p_t = \theta p_{t-1} + (1-\theta)p_t^* \qquad \Rightarrow p_t^* = \frac{1}{1-\theta} - \frac{\theta}{1-\theta}p_{t-1}$

subs. in (X): $\frac{1}{1-\theta}p_t - \frac{\theta}{1-\theta}p_{t-1} = (1-\beta\theta)s_t + \frac{\beta\theta}{1-\theta}E_t p_{t+1} - \frac{\beta\theta}{1-\theta}p_t$

with $\pi_t = p_t - p_{t-1}$: $\quad \frac{\theta}{1-\theta}\pi_t = (1-\beta\theta)s_t + \frac{\beta\theta}{1-\theta}E_t\pi_{t+1}$

to obtain the prices in terms of the output consider $s_t = (\phi+\sigma)x_t$ and add a 'cost push' shock $\tilde{u}_t$:

$$\boxed{\pi_t = \beta E_t\pi_{t+1} + \kappa x_t + \tilde{u}_t} \qquad \text{with } k = \frac{(1-\beta\theta)(1-\theta)(\phi+\sigma)}{\theta}$$

Notes:

- output gap: $x_t = y_t - y_t^n$, where the flexible prices output level can be derived from the EE:

$$y_t - \mathbf{y_t^n} = E_t[y_{t+1} - \mathbf{y_{t+1}^n}] - \frac{1}{\sigma}[i-\rho-E_t\pi_{t+1}] + E_t\Delta y_{t+1^n}$$

and then the IS curve can be alternatively set as: $x_t = E_t x_{t+1} - \frac{1}{\sigma}\left[i_t - E_t\pi_{t+1} - r_t^n\right] + \varepsilon_t$ where $r_t^n = \rho + \sigma E_t \Delta y_{t+1}^n$

- Inflation is forward looking only, has no inertia from past prices.

- The higher nominal rigidity the less sensitive the inflation to the output gap: $\uparrow \theta \rightarrow \downarrow \kappa \Rightarrow \pi$ less sensitive to x.

- cost push supply shock can be motivated by time-varying mark-up, imperfections in the labor market, labor income tax, etc.

5.3 Monetary Policy rule

- Rule vs. discretion: Discretion if time consistent leads to inflation biases (inefficient outcome) whereas a rule with commitment to time inconsistent rule leads to better outcomes.

- Discretion: each t choose optimally

- Time inconsistent: preferences or choices for action at $t+1$ is different ex-ante at t and expost at $t+1$.

Mechanism: Monetary policy is given by: $y_t = M_t - E_{t-1}\bar{P}_t$

- at $t=0$ both firms and Central Bank (CB) want low π (low M)

- at $t=1$ firms have already commited to $\bar{P}_t$ (set the period before). CB has incentive to $\uparrow M_t$ boosting Y_t

- knowing this firms anticipate higher M_t and set higher $E_{t-1}P_t \Rightarrow$ suboptimal outcome: higher π (inflationary bias)

- solution: Rule or concervative CB (Rogoff).

How to set interest rate: Instrument vs. Targeting rule

Instrument rule: use i as a policy instrument.
e.g. $i_t = (1-\gamma_i)(\alpha\pi_t + \beta x_t) + \gamma_i i_{t-1}$
Taylor rule (1993): $\boxed{i_t = \alpha + \beta_\pi(\pi_t - \pi^*) + \beta_y(y_t - y^*)}$
Taylor principle: $\beta_\pi > 1$ to ensure equilibrium, i.e. $\beta_\pi \approx 1.5, \beta_y \approx 0.5$

Targeting rule: spell out objective explicitly and then make an optimal decision

MP problem: Quadratic loss function

$$\min_{\{x_t,\pi_t\}_{t=0}^{\infty}} E_0 \textstyle\sum_{k=0}^{\infty} \beta^k \left[(\pi_{t+k} - \pi_t^*)^2 + \lambda(x_{t+k} - x_{t+k}^*)^2\right]$$
$$\text{s.t. Dynamic IS}$$
$$\text{NKPC}$$

Here π^*, x^* are the policy targets. The quadratic loss function was initially ad-hoc, however it can be derived from a second order approximation of a representative agent's UMP (i.e. is microfounded).

5.3.1 MP under discretion (No commitment)

- CB chooses $\{i_t, \pi_t, x_t\}$ to optimize the quadratic objective function. **Under discretion CB chooses the current value of the policty instrument i_t be reoptimizing each period**.

- First find optimal π_t and x_t subject to NKPC
- Use IS curve to determine i_t
- i_t supports the optimal choices of π_t, x_t

For simplicity assume: $\pi^* = 0, x^* = 0, y^* = y^n$, the optimization becomes:

$$\min_{\{x_t,\pi_t\}} \pi_t^2 + \lambda x_t^2 \quad \text{s.t.} \quad \pi_t = \kappa x_t + E_t\pi_{t+1} \text{ and Dyn. IS.}$$

Subs. π from the constraint (NKPC) and optimize, FOC:

$[x_t]:\qquad \kappa(\kappa x_t + E_t\pi_{t+1}) = -\lambda x_t \quad \Rightarrow \quad \boxed{x_t = -\frac{\kappa}{\lambda}\pi_t} \qquad (\uparrow \kappa(\downarrow \theta), \downarrow \lambda$: larger output response to inflation)

Equilibrium:

Assume a cost push process: $u_t = \rho_u u_{t-1} + \hat{u}_t$, then solve for output and inflation:

$\pi_t = \frac{\lambda}{\kappa^2+\lambda(1-\rho_u\beta)}\hat{u}_t = \lambda\phi\hat{u}_t$
$x_t = -\kappa\phi\hat{u}_t$

Derive the interest rate from the IS: $\qquad i_t = \underbrace{\frac{\lambda\rho_u+\kappa\sigma(1-\rho_u)}{\kappa^2+\lambda(1-\beta\rho_u)}}_{<1}\hat{u}_t + \sigma\tilde{g}_t$

Notice:

- The rational expectations equilibrium has the property that the CB has no incentive to change its path even if possible

- Therefore the emerging policy is denoted 'Time Consistent'

- Mechanically CB computes optimal geedback rule **taking expectations or future prices as given**

- Given optimal feedback rule, private sector forms its expectations rationally

- Taylor principle is accomplished

- Policy brings inflation back to target by pushing output gap in opposite direction

- Trade-off between variability ($Var(\pi_t)$ vs. $Var(x_t)$); trade-off depends on λ

5.3.2 Optimal MP under commitment

- Credibility and gains from commitment. The context is that one of persistent inflation bias under discretion, CB target exceeds natural output leading to inflation bias without commitment.

- Set output target $k > 0$, now: $x^* = k$

- CB chooses a plan for the path of interest rates and sticks to it forever

- With a rule, the binding commitment makes believable the policy that emerges in equilibrium

Note:

- CB **no longer takes expectations as given**, now its policy helps determine expectations

- CB now chooses **optimal sequence** of π and x subject to NKPC each period

CB solves: $\min\limits_{x_t,\pi_t} E_t \sum_{k=0}^{\infty}\beta^k\left[\pi_{t+k}^2 + \lambda(x_{t+k}-k)^2\right] \qquad$ s.t. $\pi_t = \beta E_t\pi_{t+1} + \kappa x_t + u_t$

$L = -\frac{1}{2}E_0\sum_{t=0}^{\infty}\beta^t\left(\lambda x_t^2 + \pi_t^2 + 2\gamma_t(\pi_t - \kappa x_t - \beta\pi_{t+1})\right)$

FOC:

$[x_t]:\qquad \lambda x_t - \kappa\gamma_t = 0 \qquad \forall t$

$[\pi_t]:\qquad \pi_t + \gamma_t - \gamma_{t-1} = 0$

then from the first equation at $t = 0$: $\qquad x_0 = -\frac{\kappa}{\lambda}\pi_0$

at $t = 1, 2, 3, \ldots$: $x_t = x_{t-1} - \frac{k}{\lambda}\pi_t \quad \Rightarrow \quad \boxed{\Delta x_t = -\frac{k}{\lambda}\pi_t}$

Note:

- History dependence (last equation).

- Time inconsistency (decision is not the same for all t).

Part II

Continuous time models

6 Intro: Optimization in continuous time

The framework is analogous to discrete optimization but the length of the period tends to zero. Consider the problem with a period of length $h = \frac{T}{n}$ for some $n \in \mathbb{N}$:

$$\begin{aligned} \max_{x_t, k_t} \quad & \sum_{t=0,h,2h,\ldots,T-h} hU(k_t, x_t, t) \\ \text{s.t.} \quad & k_0 = \bar{k} \\ & k_{t+h} - k_t = hA(k_t, x_t, t) \\ & k_T \geq 0 \end{aligned}$$

$$\begin{aligned} L = \sum hU(k_t, x_t, t) &+ \lambda_0(k_0 - \bar{k}) \\ &+ \lambda_h[hA(k_h, x_h, h) + k_0 - k_h] \\ &+ \lambda_{2h}[hA(k_{2h}, x_{2h}, 2h) + k_0 - k_{2h}] \\ &\vdots \\ &+ \lambda_{T+h}k_T \end{aligned}$$

FOC:

$$\begin{aligned} & h[U_x(k_t, x_t, t) + \lambda_{t+1}A_x(k_t, x_t, t)] = 0 \\ & \lambda_{t+h} - \lambda_t = -h[U_k(k_t, x_t, t) + \lambda_{t+h}A_k(k_t, x_t, t)] \\ & \lambda_T k_T = 0 \\ & \lambda_T \geq 0 \end{aligned}$$

The second equation shows that, as $h \to 0$, $\lambda_{t+h} \to \lambda_t$.

Dividing the first two equations by h and taking the limit as $h \to 0$ we get a continuous time FOC:

$U_x(k_t, x_t, t) + \lambda_t A_x(k_t, x_t, t) = 0$

Finally taking the time derivative, the continuous time Euler Equation is obtained from the second equation:

$\dot{\lambda}_t = -[U_k(k_t, x_t, t) + \lambda_t A_k(k_t, x_t, t)]$

in summary, as $h \to 0$ the continuous time model can be written as:

$\max_x \int_0^T U(k, x, t)dt$

$k_0 = \bar{k}$

$$\dot{k} = A(k, x, t)$$
$$k_T \geq 0$$

here variables moving according to dynamic differential equations as k are called *state variables*, choice variables that the agent can manipulate are called *control variables* and multipliers as λ are called *co-state variables*.

Key results:

- FOC: $U_x + \lambda A_x = 0$
- Euler Equation: $\dot{\lambda} = -[U_k + \lambda A_k]$
- Transversality condition: $\lambda_T k_T = 0$
- Non-negativity: $k_T \geq 0$

Hamiltonian

Analogous (not identical) to Lagrangian. The Hamiltonian H is given by:

$$\begin{aligned} H = U(k, x, t) + \lambda A(k, x, t) \\ H_x = 0 \\ H_k = -\dot{\lambda} \\ H_\lambda = \dot{k} \end{aligned}$$

7 Investment model (Tobin-q)

Firm PMP: $\max \int_{t=0}^{\infty} [F(K) - C(I)] e^{-rt} dt$

$H = [F(K) - C(I)] e^{-rt} + q e^{-rt} [I - \dot{K}]$

$C(I) = I + \frac{h}{2} I^2$ (cost of adjustment: $C(0) = 0, C'(0) = 1, C''(\cdot) > 0$)

FOC:

$[I]:$ $H_I = 0 \to -C'(I) e^{-rt} = -q e^{-rt} \to I = \frac{q-1}{h}$
$[K]:$ $H_k = -(\dot{qe^{-rt}}) \to F'_k = rq - \dot{q} \to \boxed{\dot{q} = qr - F'(k)}$
TVC: $\lim\limits_{t \to \infty} kqe^{-rt} = 0$

$\dot{K} = I$ then $\boxed{\dot{K} = \frac{q-1}{h}}$

Linearize dynamic system:

$$\begin{pmatrix} \dot{K} \\ \dot{q} \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{h} \\ -F''(K) & r \end{pmatrix} \begin{pmatrix} K - \tilde{K} \\ q - \tilde{q} \end{pmatrix}$$

Solve for normalized eigenvector v:

$$\begin{pmatrix} 0-\boldsymbol{\mu} & \frac{1}{h} \\ -F''(K) & r-\boldsymbol{\mu} \end{pmatrix} \begin{pmatrix} 1 \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad v = h\mu \text{ or } v = \frac{F''(k)}{r-\mu}$$

therefore solution is: $K(t) = \tilde{K} + A_1 e^{\mu_1 t}$
$q(t) = \tilde{q} + vA_1 e^{\mu_1}$

where μ_1 is the negative eigenvalue
from $K(0)$: $A_1 = K(0) - \tilde{K}$

replace $A_1 = (\tilde{K} - K(0))$ and $v = \frac{F''(K)}{r-\mu_1} = \mu_1 h$:

$$q(t) = \tilde{q} + v(\tilde{K} - K(0))e^{\mu_1 t} = \tilde{q} + v(\tilde{K} - K(t)) = \tilde{q} + (\tilde{K} - K(t))\frac{F''(K)}{r-\mu_1}$$

Phase diagram:

To get the slopes of the dynamic equations use the linearized equations at 0 (SS):

The slope of the stable arm can be obtained from the K and q solution:

$$\frac{q(t)-\tilde{q}}{K(t)-\tilde{K}} = \frac{F''(K)}{r-\mu_1} = \mu_1 h < 0$$

Slope of the $\dot{K} = 0$ line: $\dot{K} = 0 = 0(K - \tilde{K}) + \frac{1}{h}(q - \tilde{q})$ then $\frac{q-\tilde{q}}{K-\tilde{K}} = 0$

Slope of the $\dot{q} = 0$ line: $\dot{=}0 = -F''(K)(K - \tilde{K}) + r(q - \tilde{q})$ then $\frac{q-\tilde{q}}{K-\tilde{K}} = \frac{F''(K)}{r}$

Slope of stable arm vs. Slope of $\dot{q} = 0$: $\frac{F''(K)}{r-\mu_1} < \frac{F''(K)}{r}$ (stable arm is steeper)

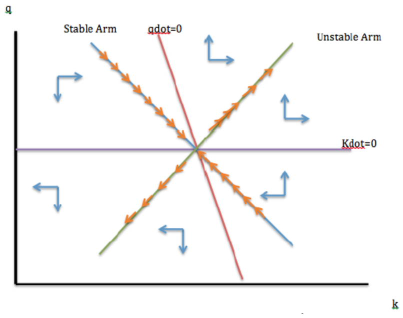


Figure 2: Phase diagram - investment model

Shocks:

Differentiate $\dot{k} = 0, \dot{q} = 0$ to get $\frac{d\tilde{q}}{dr}$ and $\frac{d\tilde{K}}{dr}$:

$$\dot{K} = 0 = \frac{q-1}{h} \Rightarrow \frac{d\tilde{q}}{d\tilde{r}} = 0$$

$$\dot{q} = 0 = qr - F'(\tilde{K}) \to \tilde{q} + \overset{0}{\cancel{\frac{d\tilde{q}}{d\tilde{r}}}} r = F''(\tilde{K})\frac{d\tilde{K}}{dr}$$

therefore: $\frac{d\tilde{K}}{dr} = \frac{\tilde{q}}{F''(\tilde{K})} < 0$

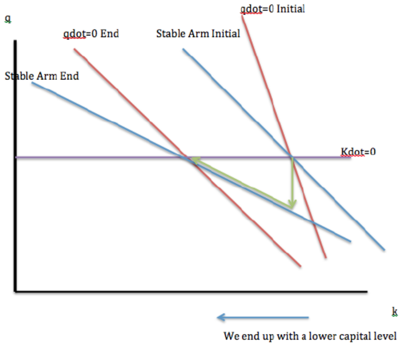


Figure 3: Effect of increase in r

7.1 Set up with labor (L):

$V(0) = \int_0^\infty [F(K,L) - wL - H(\frac{I}{K})K]e^{-rt}dt$

s.t. $\dot{K} = I$ where[5]: $C = I - \frac{h}{2}\frac{I^2}{K} = \left(\frac{I}{K} + \frac{h}{2}\frac{I^2}{K^2}K\right)$

$H = [F(K,L) - wL - H\left(\frac{I}{K}\right)K]e^{-rt} + q[I - \dot{K}]e^{-rt}$

FOC: (L: control, K: state)

$[H_L = 0]$: $F_L = w$

$[H_I = 0]$: $H'(\frac{I}{K}) = q$

$[H_K = -(\dot{qe^{-rt}})]$: $F_K - H + H'\frac{I}{K} = -\dot{q} + qr$

Assuming $F(\cdot)$ is HOD in K, L and substituting FOC:

$$\begin{aligned} F(K,L) - wL - H(\tfrac{I}{K})K &= F_kK + \cancel{F_LL} - \cancel{wL} \\ &= K[-\dot{q} + qr - q\tfrac{\dot{K}}{K}] \\ &= r(qK) - (\dot{qK}) \qquad \text{(product rule reversed in last term)} \end{aligned}$$

Then the problem becomes: $V(0) = \int_0^\infty [r(qK) - (\dot{qK})]e^{-rt}dt$

Integrating by parts: $V(0) = \int_0^\infty r(qK)e^{-rt}dt - \overset{qKe^{-rt}|_0^\infty}{\cancel{\int_0^\infty (\dot{qK})e^{-rt}}}$

---

[5] H is just the cost function of capital transformed in terms of the ratio investment to capital

8 Representative Agent model

HH UMP: $\max \int_0^\infty U(c,l,g)e^{-\beta t}dt$ s.t. $c + \dot{k} + \dot{b} = wl + rk + rb - T$

g: real government expenditure (not a choice variable)
$U_c > 0, U_{cc} < 0$
$U_g > 0, U_{gg} < 0$
$U_l < 0, U_{ll} < 0$

Agents: Households (HH), Firms, Government.
choice variables: c, l, k, b where k,b are sluggish variables (states).

$$H = U(c,l,g)e^{-\beta t} + qe^{-\beta t}\Big[F(k,l) + rb - T - c - \dot{k} - \dot{b}\Big]$$

$[H_c = 0]:$ $\quad U_c(c,l,g) = \lambda$
$[H_I = 0]:$ $\quad U_l(c,l,g) = -\lambda F_l(k,l)$
$[H_k = -\dot{(qe^{-rt})}]:$ $\quad F_k(k,l) = \beta - \frac{\dot{\lambda}}{\lambda}$
$[H_b = -\dot{(qe^{-rt})}]:$ $\quad r = \beta - \frac{\dot{\lambda}}{\lambda}$
(remember the last two come from $\lambda r = -\dot{\lambda} + \lambda\beta$)

(Important) from the first two conds: $\quad \mathbf{c} = \mathbf{c}(\lambda, \mathbf{k}, \mathbf{g})$
(system is function of λ, k) $\quad l = \mathbf{l}(\lambda, \mathbf{k}, \mathbf{g})$

Gov. BC: $\quad \dot{b} + T = g + rb$
subs. in HH BC: $\quad \mathbf{F}(\mathbf{k}, \mathbf{l}) - \mathbf{c} - \mathbf{g} = \dot{\mathbf{k}}$ (total feasibility)

The model has multiple variables, to understand the dynamics in terms of λ, k we need to get the partial effects of the other variables w.r.t. these. Taking g as given we fully differentiate the FOC:

w.r.t. $\boldsymbol{\lambda}$:

$U_{cc}\frac{\partial c}{\partial \lambda} + U_{cl}\frac{\partial l}{\partial \lambda} = 1$
$U_{lc}\frac{\partial c}{\partial \lambda} + U_{ll}\frac{\partial l}{\partial \lambda} = -F_l(k,l) - \lambda F_{ll}\frac{\partial l}{\partial \lambda}$

Matricially:

$$\begin{pmatrix} U_{cc} & U_{cl} \\ U_{lc} & U_{ll} + \lambda F_{ll} \end{pmatrix} \begin{pmatrix} \frac{\partial c}{\partial k} \\ \frac{\partial l}{\partial k} \end{pmatrix} = \begin{pmatrix} 1 \\ -F_l \end{pmatrix}$$

by Cramer rule we get:

$$c_\lambda = \frac{\begin{vmatrix} 1 & U_{cl} \\ -F_l & U_{ll} + \lambda F_{ll} \end{vmatrix}}{U_{cc}(U_{ll}+\lambda F_{ll}) - U_{cl}^2} = \frac{U_{ll}+\lambda F_{ll}+F_l U_{cl}}{Det} < 0$$

Since Det < 0 due to $U_{cl} < 0$ (MU of consumption increases in leisure, leisure (labor) and consumption are complementary (suplementary) goods).

$$l_\lambda = \frac{\begin{vmatrix} U_{cc} & 1 \\ U_{lc} & -F_l \end{vmatrix}}{U_{cc}(U_{ll}+\lambda F_{ll})-U_{cl}^2} = \frac{-U_{cc}F_l-U_{lc}}{Det} > 0$$

w.r.t. k:

$$U_{cc}\frac{\partial c}{\partial k} + U_{cl}\frac{\partial l}{\partial k} = 0$$
$$U_{lc}\frac{\partial c}{\partial k} + U_{ll}\frac{\partial l}{\partial k} = -\lambda F_{lk}(k,l) - \lambda F_{ll}(k,l)\frac{\partial l}{\partial k}$$

Matricially:

$$\begin{pmatrix} U_{cc} & U_{cl} \\ U_{lc} & U_{ll}+\lambda F_{ll} \end{pmatrix}\begin{pmatrix} \frac{\partial c}{\partial k} \\ \frac{\partial l}{\partial k} \end{pmatrix} = \begin{pmatrix} 0 \\ -\lambda F_{lk} \end{pmatrix}$$

Analogously: $c_k < 0$ and $l_k > 0$ assuming $F_{lk} > 0$

Therefore the labor and capital are complementary goods. In the same fashion if the capital increases and labor also does, consumption decreases. Intuitively that can also be thought as the direct effect on the BC of increasing the capital.

w.r.t. g:

In the same way we get:

$$\begin{pmatrix} U_{cc} & U_{cl} \\ U_{lc} & U_{ll}+\lambda F_{ll} \end{pmatrix}\begin{pmatrix} \frac{\partial c}{\partial g} \\ \frac{\partial l}{\partial g} = \end{pmatrix}\begin{pmatrix} -U_{cg} \\ -U_{lg} \end{pmatrix}$$

with $Det = U_{cc}(U_{ll}+\lambda F_{ll}) - U_{cl}^2 > 0$ (we assume the first term dominates).

The statics cannot be signed unless we know $sgn(U_{cg})$).

Dynamic Equations:

$\boxed{\dot\lambda = \lambda[\beta - F_k(k,l)]}$ gotten from FOC

$\boxed{\dot k = F(k,l) - c - g}$ (gotten from joint BC)

in SS:

$\boxed{F(\tilde k, \tilde l(\tilde\lambda, \tilde k, g)) = c(\tilde\lambda, \tilde k, g) + g}$

$\boxed{\beta = F_k(k, l(\tilde\lambda, \tilde k, g)) = r}$

Linearize the equations at SS (i.e. $\dot k, \dot\lambda = 0$):

$$\begin{pmatrix} \dot k \\ \dot\lambda \end{pmatrix} = \begin{pmatrix} F_k + F_l l_k - c_k & F_l l_\lambda - c_\lambda \\ -\lambda(F_{kk} + F_{kl} l_k) & -\lambda F_{kl} l_\lambda \end{pmatrix}\begin{pmatrix} k - \tilde k \\ \lambda - \tilde\lambda \end{pmatrix}$$

Notice the a_{22} term is $-\lambda F_{kl} l_\lambda$ instead of $\beta - F_k - \lambda F_{kl} l_\lambda$, here it is used that $\beta - F_k = 0$ in SS.

We find the standarized eigenvector associated to the negative eigenvalue:

$v = \frac{\mu_1 - a_{11}}{a_{12}} = \frac{a_{21}}{\mu_1 - a_{22}}$
$v = \frac{\mu_1 - F_k - F_l l_k + c_k}{F_l l_\lambda - c_\lambda} = \frac{-\lambda(F_{kk} + F_{kl} l_k)}{\mu_1 + \lambda F_{kl} l_\lambda}$

The solution is given by:

$$k(t) = \tilde{k} + (k(0) - \tilde{k}) e^{\mu_1 t}$$
$$\lambda(t) = \tilde{\lambda} + \mathbf{v}(k(t) - \tilde{k})$$

Phase diagram

Slope of the Stable Arm:

$\frac{\lambda(t) - \tilde{\lambda}}{k(t) - \tilde{k}} = v = \frac{\mu_1 - F_k - F_l l_k + c_k}{F_l l_\lambda - c_\lambda} = \frac{(-)}{(+)} < 0$

Note: the other version of v is consistent, either can be used to determine the sign. The other one, in this case, allows to determine the positive slope of the unstable arm, i.e., the slope of the solution with the positive eigenvalue.

Slope of the $\dot{k} = 0$ line:

$\dot{k} = 0 = a_{11}(k - \tilde{k}) + a_{12}(\lambda - \tilde{\lambda}) \quad \Rightarrow \quad \frac{\lambda - \tilde{\lambda}}{k - \tilde{k}} = \frac{a_{11}}{-a_{12}}$
$\frac{\lambda - \tilde{\lambda}}{k - \tilde{k}} = -\frac{F_k + F_l l_k - c_k}{F_l l_\lambda - c_\lambda} < 0$

Slope of the $\dot{\lambda} = 0$ line:

$\dot{\lambda} = 0 = a_{21}(k - \tilde{k}) + a_{22}(\lambda - \tilde{\lambda}) \quad \Rightarrow \quad \frac{\lambda - \tilde{\lambda}}{k - \tilde{k}} = -\frac{a_{21}}{a_{22}}$
$\frac{\lambda - \tilde{\lambda}}{k - \tilde{k}} = -\frac{-\lambda(F_{kk} + F_{kl} l_k)}{-\lambda F_{kl} l_\lambda} > 0$

Notice that here we assume that $F_{kl} l_k < F_{kk}$.

Slope of the stable arm vs. Slope of the $\dot{k} = 0$ line:

$\frac{\mu_1 - F_k - F_l l_k + c_k}{F_l l_\lambda - c_\lambda} > -\frac{F_k + F_l l_k - c_k}{F_l l_\lambda - c_\lambda} \Rightarrow$ Stable arm is steeper.

Slope of the unstable arm vs. Slope of the $\dot{\lambda} = 0$:

$-\frac{-\lambda F_{kl} l_\lambda}{\mu_2 - \lambda(F_{kk} + F_{kl} l_k)} < -\frac{-\lambda F_{kl} l_\lambda}{-\lambda(F_{kk} + F_{kl} l_k)} \quad \Rightarrow \quad$ then $\dot{\lambda} = 0$ line is steeper (more positive).

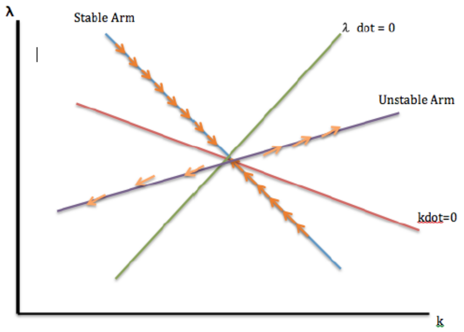


Figure 4: Effect of increase in r

Effects of an permanent increase in g: Δg

We total differentiate the SS equations (i.e. $\dot{k}, \dot{\lambda} = 0$):

$$(F_k + F_l l_k - c_k)\frac{\partial k}{\partial g} + (F_l - c_\lambda)\frac{\partial \lambda}{\partial g} = 1 + c_g - F_l l_g$$

$$(F_{kk} + F_{kl} l_k)\frac{\partial k}{\partial g} + F_{kl} l_\lambda \frac{\partial \lambda}{\partial g} = -F_{kl} l_g$$

Matricially,

$$\begin{pmatrix} F_k + F_l l_k - c_k & F_l - c_\lambda \\ F_{kk} + F_{kl} l_k & F_{kl} l_\lambda \end{pmatrix} \begin{pmatrix} \frac{\partial k}{\partial g} \\ \frac{\partial \lambda}{\partial g} \end{pmatrix} = \begin{pmatrix} 1 + c_g - F_l l_g \\ -F_{kl} l_g \end{pmatrix}$$

$$Det = (F_k + F_l l_k - c_k) F_{kl} l_\lambda - (F_{kk} + F_{kl} l_k)(F_l - c_\lambda) > 0$$

Using Cramer's method we get expressions for $\frac{\partial k}{\partial g}, \frac{\partial \lambda}{\partial g}$. Unfortunately, with no further information on l_g, c_g these cannot be signed.

However we can still use an additive separable version of the model to do so:

Additive Separable model: $U(c, l) + v(g)$

Using the same procedure we get: $\mathbf{c} = \mathbf{c}(\lambda, \mathbf{k})$
(notice g is not present) $\mathbf{l} = \mathbf{l}(\lambda, \mathbf{k})$

Then the SS equations gotten are:

$$\boxed{F(\tilde{k}, \tilde{l}(\tilde{\lambda}, \tilde{k})) = c(\tilde{\lambda}, \tilde{k}) + g}$$
$$\boxed{\beta = F_k(k, l(\tilde{\lambda}, \tilde{k}))}$$

by differentiating the equations we get (same LHS, different RHS due no to partial g effects):

$$\begin{pmatrix} F_k + F_l l_k - c_k & F_l - c_\lambda \\ F_{kk} + F_{kl} l_k & F_{kl} l_\lambda \end{pmatrix} \begin{pmatrix} \frac{\partial k}{\partial g} \\ \frac{\partial \lambda}{\partial g} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

using Cramer's rule:

$$\frac{\partial k}{\partial g} = \frac{F_{kl} l_\lambda}{Det} > 0$$
$$\frac{\partial \lambda}{\partial g} = -\frac{F_{kk} + F_{kl} l_k}{Det} > 0$$

where we assume $|F_{kk}| > F_{kl} l_k$

to get the jump in λ we differentiate the solution for $\lambda(t)$ at $t = 0$ w.r.t. g:

$\lambda(t) = \tilde{\lambda} + v(k(t) - \tilde{k})$

$\frac{\partial \lambda(0)}{\partial g} = \frac{\partial \tilde{\lambda}}{\partial g} + v(\cancelto{0}{\frac{\partial k(0)}{\partial g}} - \frac{\partial \tilde{k}}{\partial g}) > 0$

here we observe the *resource withdrawal effect:* $\Delta g \rightarrow \uparrow T \rightarrow \downarrow$ wealth $\Rightarrow \uparrow \lambda \rightarrow \uparrow L \rightarrow \uparrow k \Rightarrow \tilde{k} > 0, \tilde{\lambda} > 0$

Intertemporal BC:

before $\dot{k} = F(k, l) - c$
now $\dot{b} + \dot{k} = wl + rk + rb - T - c$
let $W \equiv b + k$, then:

$\dot{W} = r\,\overbrace{(b + k)}^{W} + wL - T - c$
$\dot{W} - rW = wL - T - c$ (diff. eq. form: the function in the LHS)
$\dot{e}^{-rt}(W - rW) = e^{-rt}(wL - T - c)$ (e^{-rt} is the integrating factor)
$\frac{dWe^{-rt}}{dt} = e^{-rt}(wL - T - c)$
$We^{-rt} = \int_0^t (wL - T - c)e^{-r\tau} d\tau + W_0$

W_0 is the constant of integration, by taking the limit as $t \rightarrow \infty$ the LHS tends to zero (assuming W doesn't blow up), therefore it must hold that:

$W_0 = -\int_0^\infty (wL - T - c)e^{-r\tau} d\tau$

subs W_0 :

$We^{-rt} = \int_0^t (wL - T - c)e^{-r\tau} d\tau - \int_0^\infty (wL - T - c)e^{-r\tau} d\tau$
$W = -[\int_t^\infty (wL - T - c)e^{-r\tau} d\tau]e^{rt}$

replace the consumption function $\mathbf{c(\tau) = c(t)e^{(r-\beta)(\tau - t)}}$

$W = -e^{rt} \int_t^\infty \left(wl - T - c(t)e^{(r-\beta)(\tau - t)}\right) e^{-r\tau} d\tau$

solve for $c(t)$: $\boxed{c(t) = \frac{b(t) + k(t) + e^{rt} \int_t^\infty (wl - T)e^{-r\tau} d\tau}{e^{rt} \int_t^\infty e^{-rt - \beta(\tau - t)}}}$

Note: The consumption funtion replaced comes from assuming an additive separable utility function of the form $U = \ln(c) + \ln(l)$ and manipulating its FOC:

$\dot{\lambda} = \lambda[\beta - r]$
$\frac{1}{c} = \lambda$

$\dot{(\frac{1}{c})} = \dot{\lambda}$
$-\frac{1}{c^2}\dot{c} = \dot{\lambda}$
$-\frac{1}{\lambda}\frac{1}{c^2}\dot{c} = \frac{\dot{\lambda}}{\lambda}$
$-\frac{\dot{c}}{c} = \frac{\dot{\lambda}}{\lambda}$

using the other FOC:

$$\dot{c} = c[r-\beta]$$
$$\dot{c} - c[r-\beta]$$
$$\frac{d}{dt}\left(c(t)e^{-(r-\beta)t}\right) = 0$$
$$c(t)e^{-(r-\beta)t} = c_0 \quad \Rightarrow \quad c(t) = c_0 e^{(r-\beta)t}$$

then: $c(\tau) = c(t)e^{(r-\beta)(\tau-t)}$

8.1 Rep. Agent model applications

Optimal Taxation

- Set with an additive separable utility for simplicity. HH optimization with taxes:

$$H = [U(c,l)+V(g)]e^{-\beta t} + \lambda e^{-\beta t}[r(1-\tau_k)k + w(1-\tau_w)l - c - \dot{k}]$$

FOC:

$[c]: \quad U_c = \lambda$

$[l]: \quad U_l = -\lambda F_l = -\lambda w(1-\tau_w)$

$[k]: \quad \mathbf{r(1-\tau_k)} = \beta - \frac{\dot{\lambda}}{\lambda} \qquad$ in SS: $\boxed{r(1-\tau_k) = \beta} \qquad (\star_1)$

Getting the implementability constraint: We want to get a BC consistent with an optimizing behaviour that considers taxes. In order to do that we replace in the regular BC the FOC of the HH UMP with taxes. That will be a constraint of the central planner afterwards.

$$\dot{k} = (\beta - \tfrac{\dot{\lambda}}{\lambda})k - \tfrac{U_l}{\lambda}l - c$$

multiply by λ:

$$\lambda\dot{k} + \dot{\lambda}k = \lambda\beta k - U_l l - \lambda c$$

Define[6] $\mu \equiv U_c k$,

$\boxed{\dot{\mu} = \beta\mu - U_l l - U_c c}$ [IC: Implementability constraint]

Central Planner optimization:

The central planner tries to get the best possible result maximizing the HH utility subject to both MC and the IC:

$$H = [U(c,l)+V(g)]e^{-\beta t} + S_1 e^{-\beta t}[F(k,l) - c - \dot{k}] + S_2 e^{-\beta t}[\beta\mu - U_l l - U_c c - \dot{\mu}] + \theta e^{-\beta t}[U_c k - \mu]$$

FOC:

$[c]: \quad U_c - S_1 - S_2[U_{lc}l + U_c + U_{cc}c] + \theta U_{cc}k = 0$

$[l]: \quad U_l + S_1 F_l - S_2[U_{ll}l + U_l + Uclc] + \theta U_{cl}k = 0$

---

[6] new accumulative variable that mimics optimization with taxes

$$[k]: \qquad S_1 F_k + \theta U_c = S_1\beta - \dot{S}_1$$
$$[\mu]: \qquad S_2\beta - \theta = \dot{S}_2 - \beta S_2 \quad \rightarrow \quad \boxed{\dot{S}_2 = \theta} \Rightarrow \text{in SS: } \theta = 0$$

in the equilibrium at the SS:

$$\dot{k} = 0 \quad \Rightarrow \quad F(\tilde{k}, l) = \tilde{c}$$
$$\dot{S}_2 = 0 \quad \Rightarrow \quad \tilde{\theta} = 0$$
$$\dot{S}_1 = 0 \quad \Rightarrow \quad \tilde{S}_1 F_k(\tilde{k}, l) + \tilde{\theta} U_c = \tilde{S}_1 \beta \quad \Rightarrow \quad \boxed{F_k(\tilde{k}, l) = \beta} \qquad (\star_2)$$

$(\star_1)$ and $(\star_2)$ are compatible iff $\tau_{\mathbf{k}} = \mathbf{0}$ (then the optimal allocation implies not taxing the capital).

Finally with this result the MC condition becomes:

$\dot{k} = rk + w(1-\tau_w)l - c$ and in SS: $rw + wl - wl\tau_w - c \quad \rightarrow \quad \tau_w = \frac{g}{wl}$ or rather said $g = wl\tau_w$, i.e., the public expenditure should be funded entirely by taxes on labor.

Bond - Equity financing

Capital is financed only by bonds (public, private) and equity. In the following model s is the price of equity stocks, b_p, b_g are private and public bonds, τ_c are taxes on capital/stocks gains and τ_p are taxes on profits.

HH UMP: max $\quad \int_0^\infty U(c,l,m,g)e^{-\beta t}dt$
s.t. $c + \dot{b}_g + \dot{b}_p + \dot{m} + s\dot{E} = wl + r_g b_g + r_p b_p - p(b_g + b_p + m) + isE - T_h$

where $T_h = \underbrace{\tau_y[wl - r_g b_g + r_p b_b + isE]}_{\text{taxes on income}} + \quad \underbrace{\tau[\dot{s} + s\theta]E}_{\text{taxes on capital gains}}$ (Taxes paid by firms)
also: $T_f = \tau_p[y - wl - r_p b_p]$ (Taxes paid by firms)

Bond financing:

FOCs of Firm and HH optimization problem wrt b_p evaluated at SS:

$$\underbrace{(1-\tau_p)f_k(k)}_{\text{tax deduced returns on capital}} = \underbrace{r_p(1-\tau_p)}_{\text{returns paid to bond holders}} - \underbrace{p}_{\text{inflation: in SS}\rightarrow\frac{\dot{M}}{M}=\theta=p} \qquad \text{(Firm)}$$

$$r_p(1-\tau_y) - \underbrace{p}_{\theta} = \beta \qquad \text{(HH)}$$

From the HH condition: $\qquad r_p = \frac{\theta+\beta}{1-\tau_y}$

subst. in the Firm condition: $\qquad \boxed{(1-\tau_p)f_k(k) = \beta\left(\frac{1-\tau_p}{1-\tau_y}\right) + \frac{\theta(\tau_y-\tau_p)}{1-\tau_y}} \qquad (\clubsuit)$

if the last term of the RHS is positive depends on the sign of $\tau_y - \tau_p$. We will see that depending on the magnitude of taxes either form of financing can be the optimal one.

Financing with equities:

$(1-\tau_p)f_k(k) = \bar{i} + \frac{\dot{s}}{s}$ (Firms)

$\bar{i}(1-\tau_y) + \frac{\dot{S}}{S} - \tau_c\theta = \beta$ (HH)

here: $S = sp$ and $\frac{\dot{S}}{S} = \frac{\dot{s}}{s} + p = \frac{\dot{s}}{s} + \theta$

then:

$\bar{i}(1-\tau_y) + (1-\tau_c)\frac{\dot{s}}{s} - \tau_c\theta = \beta$ (HH)

rearrange: $\bar{i} + \frac{\dot{s}}{s} = \frac{\beta+\tau_c\theta}{1-\tau_c} + \frac{\bar{i}(\tau_y-\tau_c)}{1-\tau_c}$

subs. in the firms FOC: $\boxed{(1-\tau_p)f_k(k) = \frac{\beta+\tau_c\theta}{1-\tau_c} + \frac{\bar{i}(\tau_y-\tau_c)}{1-\tau_c}}$ (♠)

Miller-Modigliani result: Notice that without taxes we get in each case that $f_k(k) = \beta$, i.e. the marginal product of capital is equal as the discount factor as expected. That means that with no distortions how the firms are financed is irrelevant.

Super neutrality

with bond financing: we take the derivative of (♣):

$$\frac{dk}{d\theta} = \frac{1}{(1-\tau_p)f_{kk}}\frac{\tau_y-\tau_p}{1-\tau y}$$

with equity financing: we take the derivative of (♠):

$$\frac{dk}{d\theta} = \frac{1}{(1-\tau_p)f_{kk}}\frac{\tau_c}{1-\tau_c}$$

Therefore money is not neutral in either case and withouth taxes we have that money is neutral ($\frac{dk}{d\theta} = 0$)

Heterogeneity in the Representative agent model

The source of heterogeneity in this model is the endowments of capital:

$\dot{k}_i = rk_i + w - c_i$

Notice the labor provided per agent is 1.

$H = \frac{1}{\gamma}c_i^{\gamma}e^{-\beta t} + \lambda_i e^{-\beta t}[rk_i - w - c_i - \dot{k}_i]$

FOC:

$[c]: \quad c_i^{\gamma-1} = \lambda_i$

$[k]: \quad r = \beta - \frac{\dot{\lambda}}{\lambda}$

then, $(\gamma-1)c_i^{\gamma-2}\dot{c}_i = \dot{\lambda}_i$

$\frac{1}{c_i^{\gamma-1}}(\gamma-1)c_i^{\gamma-2}\dot{c}_i = \frac{1}{\lambda}\dot{\lambda}_i$

$\frac{\dot{c}_i}{c_i} = \frac{\dot{\lambda}}{\lambda}\frac{1}{(\gamma-1)} = \frac{r-\beta}{1-\gamma} = \frac{\dot{c}}{c}$ (constant)

since $\frac{\dot{c}_i}{c_i}$ is constant (the consumption grows at a constant rate):

$c_i(t)e^{-ct} = c_{i0}$ and in aggregate terms: $c(t)e^{-ct} = c_0$, then the individual consumption function is:

$\boxed{c_i(t) = \frac{c(t)}{L}\frac{\mathbf{c_{i0}}}{\mathbf{c_0/L}}}$ $\Rightarrow$ the consumption depends on the relative initial endowment with respect to the population average ($\theta = \frac{c_{i0}}{c_0/L}$).

9 Sidrausky model: Money in Utility

- Money included in utility
- As before, the model is set in real terms but with money holdings involved, the BC is set in terms of the inflation rate, this rate is present in the dynamics of the real money and bond holdings.

Nominal BC → Real BC

$$\dot{B} + Pc + P\dot{k} + \dot{M} = PF(k,l) + iB - PT$$
$$\tfrac{\dot{B}}{P} + c + \dot{k} + \tfrac{\dot{M}}{P} = F(k,l) + \tfrac{iB}{P} - T$$
$$\tfrac{\dot{B}}{P} + c + \dot{k} + \tfrac{\dot{M}}{P} = F(k,l) + \tfrac{iB}{P} - T$$

Now consider real money/bond holdings dynamics:

$$\left(\tfrac{\dot{M}}{P}\right) = \tfrac{\dot{M}P - M\dot{P}}{P^2} \Rightarrow \dot{m} = \tfrac{\dot{M}}{P} - m\cancelto{\rho}{\tfrac{\dot{P}}{P}} \to \tfrac{\dot{M}}{P} = \dot{m} + m\rho$$

then,

$$\dot{b} + b\rho + c + \dot{k} + \dot{m} + m\rho = F(k,l) + ib - T$$
$$\dot{b} + c + \dot{k} + \dot{m} + m\rho = F(k,l) + \underbrace{(i-\rho)}_{r} b - T$$

$$\boxed{\dot{b} + c + \dot{k} + \dot{m} + m\rho = F(k,l) + rb - T} \qquad \text{(Real BC)}$$

The UMP is the following:

$$H = U(c,l,m,g)e^{-\beta t} + \lambda e^{-\beta t}(F(k,l) + rb - \rho m - c - T - \dot{k} - \dot{b} - \dot{m})$$

FOC: (c, l: control; m, k, b: state. FOC: $H_{control} = 0, H_{state} = -(\dot{\lambda e^{-\beta t}})$)

$[c]: \quad U_c(c,l,m,g) = \lambda$
$[l]: \quad U_l(c,l,m,g) = -\lambda F_l(k,l)$
$[m]: \quad \frac{U_m(c,l,m,g)}{\lambda} - \rho = \beta - \frac{\dot{\lambda}}{\lambda}$
$[k]: \quad F_k(k,l) = \beta - \frac{\dot{\lambda}}{\lambda}$
$[b]: \quad r = \beta - \frac{\dot{\lambda}}{\lambda}$

From these equations we state that:

$c = c(\lambda, k, m, g)$
$l = l(\lambda, k, m, g)$
$\rho = \rho(\lambda, k, m, g)$

Therefore the dynamic equations driving the system are: k, λ, m

Goods market clearing conditions: (Gov BC + HH BC)

Goverment Nominal BC: $\dot{B} + \dot{M} = iB - PT + Pg$
real BC: $\dot{b} + b\rho + \dot{m} + m\rho = ib - T - g$

subs in HH BC:

$$\dot{k} + c + \cancel{rb} - \cancel{T} - g = F(k,l) + \cancel{rb} - \cancel{T}$$
$$\boxed{\dot{k} = F(k,l) - c - g}$$

from the FOCs: $\boxed{\dot{\lambda} = \lambda(\beta - F_k(k,l))}$

Finally letting $\frac{\dot{M}}{M} = \phi$ (constant growth of money) and rearranging the expression for $\frac{\dot{M}}{P}$:

$$\frac{\dot{M}}{P} = \dot{m} + m\rho$$
$$\frac{P}{M}\frac{\dot{M}}{P} = \frac{P}{M}\dot{m} + \frac{P}{M}\frac{M}{P}\rho$$
$$\phi = \frac{\dot{m}}{m} + \rho$$

then: $\boxed{\dot{m} = m(\phi - \rho)}$

then the dynamic equations characterizing the system of variables are given by:

$$\dot{k} = F(k, l(\lambda, k, m, g)) - c(\lambda, k, m, g) - g$$
$$\dot{\lambda} = \lambda(\beta - F_k(k, l(\lambda, k, m, g)))$$
$$\dot{m} = m(\phi - \rho(\lambda, k, m, g))$$

In SS:

$$\mathbf{F}(\tilde{\mathbf{k}}, \mathbf{l}(\tilde{\lambda}, \tilde{\mathbf{k}}, \tilde{\mathbf{m}}, \mathbf{g})) = \mathbf{c}(\tilde{\lambda}, \tilde{\mathbf{k}}, \tilde{\mathbf{m}}, \mathbf{g}) + \mathbf{g}$$
$$\mathbf{F_k}(\tilde{\mathbf{k}}, \mathbf{l}(\tilde{\lambda}, \tilde{\mathbf{k}}, \tilde{\mathbf{m}}, \mathbf{g})) = \boldsymbol{\beta}$$
$$\boldsymbol{\phi} = \boldsymbol{\rho}(\tilde{\lambda}, \tilde{\mathbf{k}}, \tilde{\mathbf{m}}, \mathbf{g})$$

Linearization:

$$\begin{pmatrix} \dot{k} \\ \dot{\lambda} \\ \dot{m} \end{pmatrix} = \begin{pmatrix} F_k + F_l l_k - c_k & F_l l_\lambda - c_\lambda & \mathbf{F_l l_m - c_m} \\ -\tilde{\lambda}(F_{kk} + F_{kl} l_k) & -\tilde{\lambda}(F_{kl} l_\lambda) & -\tilde{\lambda}(\mathbf{F_{kl} l_m}) \\ -\tilde{m}\rho_k & -\tilde{m}\rho_\lambda & -\tilde{m}\rho_m \end{pmatrix} \begin{pmatrix} k - \tilde{k} \\ \lambda - \tilde{\lambda} \\ m - \tilde{m} \end{pmatrix}$$

Effects of money in the LR are captured by the elements $a_{13}, a_{2,3}$ in this matrix. Hence we can see that in the standard non-separable utility case the money is not neutral in the LR.

Additively separable utility:

$$H = [U(c,l,g) + V(m)]e^{-\beta t} + \lambda e^{-\beta t}[F(k,l) + rb - c - \rho m - T - \dot{k} - \dot{m} - \dot{b}]$$

FOC:

$$\begin{array}{ll}
[c]: & U_c(c,l,g) = \lambda \\
[l]: & U_l(c,l,g) = -\lambda F_k(k,l) \\
[m]: & \frac{V'(m)}{\lambda} - \rho = \beta - \frac{\dot{\lambda}}{\lambda} \\
[k]: & F_k(k,l) = \beta - \frac{\dot{\lambda}}{\lambda} \\
[b]: & r = \beta - \frac{\dot{\lambda}}{\lambda}
\end{array}$$

therefore the system is solved for:

$$\begin{array}{l}
c = c(\lambda, k, g) \\
l = l(\lambda, k, g) \\
\rho = \rho(\lambda, k, m, g)
\end{array}$$

See how money doesn't appear in the SR demands.

The dynamic equations are:

$$\begin{array}{l}
\dot{k} = F(k, l(\lambda, k, g)) - c(\lambda, k, g) - g \\
\dot{\lambda} = \lambda[\beta - F_k(k, l(\lambda, k, g))] \\
\dot{m} = m[\phi - \rho(\lambda, k, m, g)]
\end{array}$$

Equilibrium (SS):

$$\begin{array}{l}
F(\tilde{k}, l(\tilde{\lambda}, \tilde{k}, g)) = c(\tilde{\lambda}, k, g) + g \\
F_k(\tilde{k}, l(\tilde{\lambda}, \tilde{k}, g)) = \beta \\
\phi = \rho(\tilde{\lambda}, \tilde{k}, \tilde{m}, g)
\end{array}$$

In the first two equations is seen how the real part of the economy does not respond to m, effects as $\frac{d\tilde{k}}{dg}$ will not depend on m.

the linearized matrix is:

$$\begin{pmatrix} \dot{k} \\ \dot{\lambda} \\ \dot{m} \end{pmatrix} = \begin{pmatrix} F_k + F_l l_k - c_k & F_l l_\lambda - c_\lambda & \mathbf{0} \\ -\tilde{\lambda}(F_{kk} + F_{kl} l_k) & -\tilde{\lambda}(F_{kl} l_\lambda) & \mathbf{0} \\ -\tilde{m}\rho_k & -\tilde{m}\rho_\lambda & -\tilde{m}\rho_m \end{pmatrix} \begin{pmatrix} k - \tilde{k} \\ \lambda - \tilde{\lambda} \\ m - \tilde{m} \end{pmatrix}$$

we can see that money is super neutral ("super" because in the linearized matrix we are seeing the effects of the deviations/variations of m in the dynamics of the system).

Multiplicately separable utility:

$$H = [U(c,l,g)V(m)]e^{-\beta t} + \lambda e^{-\beta t}[F(k,l) + rb - \rho m - c - T - \dot{k} - \dot{b} - \dot{m}]$$

FOC:

$[c]: \quad U_c(c,l,g)V(m) = \lambda$
$[l]: \quad U_l(c,l,g)V(m) = -\lambda F_k(k,l)$
$[m]: \quad \frac{U(c,l,g)V'(m)}{\lambda} - \rho = \beta - \frac{\dot{\lambda}}{\lambda}$
$[k]: \quad F_k(k,l) = \beta - \frac{\dot{\lambda}}{\lambda}$
$[b]: \quad r = \beta - \frac{\dot{\lambda}}{\lambda}$

the SR equations have the form:

$c = c(\lambda, k, g, m)$
$l = l(\lambda, k, g, m)$
$\rho = \rho(\lambda, k, m, g)$

The consumption and labor are functions of money. Then the transition path (linearized matrix) will be affected by money.

However notice how the MRS evaluated at the steady state does not depend m: $\frac{U_c(\tilde{c},\tilde{l},g)V(\tilde{m})}{U_l(\tilde{c},\tilde{l},g)V(\tilde{m})} = -\frac{1}{F_l(\tilde{k},\tilde{l})}$

Then in the LR labor, and consumption don't depend on m: $l = l(\tilde{\lambda}, \tilde{k}, g), \quad c = c(\tilde{\lambda}, \tilde{k}, g)$

Additionally the dynamic equations in the SS don't depend on m:

$F(\tilde{k}, l(\tilde{\lambda}, \tilde{k}, g)) = c(\tilde{\lambda}, \tilde{k}, g) + g$
$\beta = F_k(\tilde{k}, l(\tilde{\lambda}, \tilde{k}, g))$

Non-separable utility with inelastic labor:

$H = [U(c,m,g)]e^{-rt} + \lambda e^{-rt}[F(k,l) + rb - \rho m - c - T - \dot{k} - \dot{b} - \dot{m}]$

FOC:

$[c]: \quad U_c(c,m,g) = \lambda$
$[m]: \quad \frac{U_m(c,m,g)}{\lambda} = \beta - \frac{\dot{\lambda}}{\lambda}$
$[k]: \quad F_k(k) = \beta - \frac{\dot{\lambda}}{\lambda}$
$[b]: \quad r = \beta - \frac{\dot{\lambda}}{\lambda}$

then the SR optimal conditions are solved for:

$c = (\lambda, k, m, g)$
$\rho = \rho(\lambda, k, m, g)$

The dynamic equations are:

$\dot{k} = F(k) - c(\lambda, m, k, g) - g$
$\dot{\lambda} = \lambda(\beta - F_k(k))$
$\dot{m} = m(\phi - \rho(\lambda, k, m, g))$

the LR equilibrium is given by:

$F(\tilde{k}) = c(\tilde{\lambda}, \tilde{k}, \tilde{m}, g) + g$
$F_k(\tilde{k}) = \beta$
$\phi = \rho(\tilde{\lambda}, \tilde{k}, \tilde{m}, g)$

therefore with inelastic labor money could impact SR and the transition path, however it doesn't have effects in the LR steady state. This is observed in the fact that β determines the marginal product of capital F_k which determines the capital level and by feasibility the subsequent consumption level (provided g doesn't change as stated before).

Summary:

Case	**Utility f/n**	**Effect of money**
General, non-separable	$U(c, l, m, g)$	Money is not neutral, nor super neutral
Additive separable	$U(c, l, g) + V(m)$	Money is neutral and super neutral (both in transition path and SS)
Multiplicatively separable	$U(c, l, g)V(m)$	Money affects SR and transition path but money is super neutral in the LR
Non separable with inelastic labor	$U(c, g, m)$	Money can impact SR and transition path but won't affect LR (in LR marginal product of k is determined only by β, setting k at a certain level, hence doing the same with c).

10 OLG models (two periods and continuous)

Diamond

HH UMP: $\max_{c_t, c_{t+1}} U(c_t) + \frac{1}{1+\theta} U(c_{t+1})$
s.t $\quad c_t + s_t = w_t$
$\quad c_{t+1} = (1 + r_{t+1})s_t$

The lifetime BC is: $c_t + \frac{c_{t+1}}{1+r_{t+1}} = w_t$

then the problem to solve is:

$L = U(c_t) + \frac{1}{1+\theta} U(c_{t+1}) + \lambda \left[w_t - c_t + \frac{c_{t+1}}{1+r_{t+1}} \right]$

$[c_t]: \quad U'(c_t) = \lambda$
$[c_{t+1}]: \quad \frac{1}{1+\theta} U'(c_{t+1}) = \frac{\lambda}{1+r_{t+1}}$

the EE is: $\boxed{\mathbf{U'(c_t) = \frac{1+r_{t+1}}{1+\theta} U'(c_{t+1})}}$

Firm PMP: $\max \pi = F(K,N) - rK - wN$

using HDO property:

$F(K,N) = Nf(k)$
$F_K = Nf'(k)\frac{1}{N} = f'(k)$
$F_N = f(k) + Nf'(k)\frac{K}{N^2}(-1) \rightarrow F_N = f(k) - f'(k)k$

Then: $\boxed{r = f'(k)}$ and $\boxed{w = f(k) - f'(k)k}$

Capital market:

$K_{t+1} = N_t s_t$
$\frac{K_{t+1}}{N_t}\frac{N_{t+1}}{N_{t+1}} = s_t \quad \Rightarrow \quad \boxed{\mathbf{k_{t+1} = \frac{s_t}{1+n}}}$

To go further assume an specific utility function: $U = \ln(\cdot)$

$L = \ln(c_t) + \frac{1}{1+\theta}\ln(c_{t+1}) + \lambda[w_t - c_t - \frac{c_{t+1}}{1+r_{t+1}}]$

$[c_t]: \quad \frac{1}{c_t} = \lambda$
$[c_{t+1}]: \quad \frac{1}{1+\theta}\frac{1}{c_{t+1}} = \frac{\lambda}{1+r_{t+1}}$

Inverting the EE: $c_t = \frac{1+\theta}{1+r_{t+1}} c_{t+1} \qquad (+)$

Also assume a CD production function:

$\max \pi = AK^{\alpha}N^{1-\alpha} - rK - wN$

$f(k) = Ak^{\alpha}$ and then,
$r = f'(k) = \alpha A k^{\alpha-1}$
$w = (1-\alpha)Ak^{\alpha}$

use the (+) in the lifetime BC:

$\frac{1+\theta}{1+r_{t+1}} c_{t+1} + \frac{c_{t+1}}{1+r_{t+1}} = w_t$

$c_{t+1} = \frac{1+r_{t+1}}{2+\theta}(1-\alpha)Ak_t^{\alpha}$ and $c_t = \frac{1+\theta}{2+\theta}(1-\alpha)Ak_t^{\alpha}$

also savings are given by:

$s_t = w_t - c_t = w_t - \frac{1+\theta}{2+\theta}w_t \quad \Rightarrow \quad s_t = \frac{1}{2+\theta}(1-\alpha)Ak_t^{\alpha}$

the capital is:

$$k_{t+1} = \frac{s_t}{1+n} \quad \Rightarrow \quad k_{t+1} = \frac{(1-\alpha)}{1+n}\frac{1}{2+\theta}Ak_t^{\alpha}$$

Additionally at the SS: $k_{t+1} = k_t = \tilde{k}$

$$\tilde{k} = \frac{(1-\alpha)}{1+n}\frac{1}{2+\theta}A\tilde{k}^{\alpha}$$

$$\Rightarrow \qquad \tilde{k} = \left[\frac{A(1-\alpha)}{(1+n)(1+\theta)}\right]^{\frac{1}{1-\alpha}}$$

10.1 Social Security schemes

PayGo

HH UMP: $\max\limits_{c_t,c_{t+1}} U(c_t) + \frac{1}{1+\theta}U(c_{t+1})$

s.t $c_t + s_t + \boldsymbol{\tau} = w_t$

$c_{t+1} = (1+r_{t+1})s_t + (\mathbf{1}+\mathbf{n})\boldsymbol{\tau}$

subs. s_t in period one BC,

$$c_t + \frac{c_{t+1}-(1+n)\tau}{1+r_{t+1}} = w_t - \tau$$

$$c_t + \frac{c_{t+1}}{1+r_{t+1}} = w_t + \frac{\tau(n-r_{t+1})}{1+r_{t+1}} \qquad \text{[Lifetime BC]}$$

assume $U = \ln(\cdot)$:

$$[c_t]: \qquad \frac{1}{c_t} = \lambda$$

$$[c_{t+1}]: \qquad \frac{1}{1+\theta}\frac{1}{c_{t+1}} = \frac{\lambda}{1+r_{t+1}}$$

as before: $\quad c_t = \frac{1+\theta}{1+r_{t+1}}c_{t+1} \quad (+)$

subs. for c_{t+1} in the lifetime BC:

$$c_t + \frac{1}{1+r_{t+1}}\underbrace{\frac{1+r_{t+1}}{1+\theta}c_t}_{c_{t+1}} = w_t + \frac{\tau(n-r_{t+1})}{1+r_{t+1}}$$

then,

$$c_t = \left(w_t + \frac{\tau(n-r_{t+1})}{1+r_{t+1}}\right)\left(\frac{1+\theta}{2+\theta}\right)$$

$$c_{t+1} = \left(w_t + \frac{\tau(n-r_{t+1})}{1+r_{t+1}}\right)\left(\frac{1+r_{t+1}}{2+\theta}\right)$$

the savings are:

$$s_t = w_t - \tau - c_t = w_t - \tau - \left(w_t + \frac{\tau(n-r_{t+1})}{1+r_{t+1}}\right)\left(\frac{1+\theta}{2+\theta}\right)$$

and the effect of the social security fee in the savings is:

$\frac{\partial s_t}{\partial \tau} = -1 - \frac{(n-r_{t+1})}{1+r_{t+1}}\left(\frac{1+\theta}{2+\theta}\right) < 0$ Savings and SS are substitutes

Fully Funded

$$\text{HH UMP:} \quad \max_{c_t, c_{t+1}} U(c_t) + \frac{1}{1+\theta} U(c_{t+1})$$
$$\text{s.t} \quad c_t + s_t + \mathbf{b_t} = w_t$$
$$c_{t+1} = (\mathbf{1} + \mathbf{r_{t+1}})(\mathbf{s_t} + \mathbf{b_t})$$

the EE inverted is: $c_t = \frac{1+\theta}{1+r_{t+1}} c_{t+1}$

subs. from the periodwise BC: $c_t = w_t - s_t - b_t$ and $c_{t+1} = (1 + r_{t+1})(s_t + b_t)$,

$$w_t - s_t - b_t = \frac{1+\theta}{\cancel{1+r_{t+1}}}\cancel{(1+r_{t+1})}(s_t + b_t)$$
$$w_t = (2+\theta)(s_t + b_t)$$

$$\boxed{s_t + b_t = \frac{w_t}{2+\theta}}$$

before (w/o SS): $s_t = \frac{w_t}{2+\theta}$

therefore if b_t <savings w/o SS (compulsory savings are lower than initially planned savings) then the agent will choose the same aggregate savings as before.

Otherwise the SS system creates a distortion.

Summary

Scheme	**Return**	**Decision**
PayGo	$1+n$	if $r > n$ the capital generated using PayGo is lesser.
Fully Funded	$1+r_{t+1}$	

10.2 Blanchard (1985) (adding demographic features)

1. *Setup*

Cohorts are born at period v, there is a constant probability of death at each period p. Then the relevant discount factor is $e^{-\theta(t-v)}e^{-p(t-v)} = \mathbf{e}^{-(\mathbf{p}+\theta)(\mathbf{t}-\mathbf{v})}$.

Notice that $1 - \int_s^t pe^{-p(\tau-s)}d\tau = e^{-p(t-v)}$ is the probability of being alive at period t.

$$H = \ln(c(v,t))e^{-(p+\theta)(t-v)} + \lambda(v,t)e^{-(p+\theta)(t-v)}\left[y(v,t) + (r(t)+\mathbf{p})w(v,t) - c(v,t) - w_t(v,t)\right]$$

w is the wealth (sluggish variable), w_t the time derivative and the probability of death is p, the latter enters in the BC as a premium multiplying the wealth.

FOC: $(H_c = 0, \quad H_w = -\dot{\mu})$

$$[c]: \qquad \tfrac{1}{c(v,t)} = \lambda(v,t)$$
$$[w]: \quad \lambda_t(v,t) = \lambda(v,t)[\theta - r(t)]$$

$$[TVC]: \quad \lim_{t\to\infty} \lambda(v,t)w(v,t)e^{-\int_v^t (r(s)+p)ds} = 0$$

2. *Consumption growth:*

$$-\tfrac{1}{c(v,t)}\tfrac{c_t(v,t)}{c(v,t)} = \lambda_t(v,t)$$
$$-\tfrac{c_t(v,t)}{c(v,t)} = \tfrac{\lambda_t(v,t)}{\lambda(v,t)} \quad \Rightarrow \quad c_t(v,t) = [r(t)-\theta]c(v,t)$$

therefore: $\boxed{c(v,\tau) = c(v,t)e^{\int_t^\tau [r(s)-\theta]ds}}$

3. *Intertemporal Budget Constraint:*

$$w_t(v,t) = y(v,t) + (r(t)+p)w(v,t) - c(v,t)$$

$$(w_t(v,t) - (r(t)+p)w(v,t))e^{-\int_v^t [r(s)+p]ds} = (y(v,t) - c(v,t))e^{-\int_v^t [r(s)+p]ds}$$

$$\tfrac{d}{dt}\left(w(v,t)e^{-\int_v^t [r(s)+p]ds}\right) = (y(v,t) - c(v,t))e^{-\int_v^t [r(s)+p]ds}$$

$$w(v,t)e^{-\int_v^t [r(s)+p]ds} = \int_v^t \left(y(v,\tau) - c(v,\tau)\right)e^{-\int_v^\tau [r(s)+p]ds}d\tau + w_0$$

Apply TVC to the LHS and get RHS = 0 as $t \to \infty$, then:

$$w_0 = -\int_v^\infty \left(y(v,\tau) - c(v,\tau)\right)e^{-\int_v^\tau [r(s)+p]ds}d\tau$$

$$w(v,t)e^{-\int_v^t [r(s)+p]ds} = -\int_t^\infty \left(y(v,\tau) - c(v,\tau)\right)e^{-\int_v^\tau [r(s)+p]ds}d\tau$$

then:

$$w(v,t) = -\int_t^\infty \left(y(v,\tau) - c(v,\tau)\right)e^{-\int_t^\tau [r(s)+p]ds}d\tau$$

$$w(v,t) + \underbrace{\int_t^\infty y(v,\tau)e^{-\int_t^\tau [r(s)+p]ds}d\tau}_{h(v,t)} = \int_t^\infty c(v,\tau)e^{-\int_t^\tau [r(s)+p]ds}d\tau$$

$$w(v,t) + h(v,t) = \int_t^\infty c(v,\tau)e^{-\int_t^\tau [r(s)+p]ds}d\tau$$

subs. $c(v,\tau) = c(v,t)e^{\int_t^\tau [r(s)-\theta]ds}$

$$w(v,t) + h(v,t) = \int_t^\infty c(v,t)e^{\int_t^\tau [r(s)-\theta]ds}e^{-\int_t^\tau [r(s)+p]ds}d\tau$$

$$w(v,t) + h(v,t) = c(v,t)\int_t^\infty e^{\int_t^\tau -[p+\theta]ds}d\tau$$

$$w(v,t) + h(v,t) = c(v,t)\tfrac{1}{p+\theta}$$

$$\boxed{c(v,t) = (p+\theta)[w(v,t) + h(v,t)]}$$

4. *Aggregation of variables:*

- The aggregation is done through cohorts' birth dates v

The size of population is: $P(t) = \int_{-\infty}^{t} pe^{-p(t-v)}dv$

in the same fashion the rest of variables are aggregated:

$C(t) = \int_{-\infty}^{t} c(v,t)pe^{-p(t-v)}dv$

$W(t) = \int_{-\infty}^{t} w(v,t)pe^{-p(t-v)}dv$

$H(t) = \int_{-\infty}^{t} h(v,t)pe^{-p(t-v)}dv = \int_{-\infty}^{t}\int_{t}^{\infty} y(v,\tau)e^{-\int_t^\tau [r(s)+p]ds}d\tau pe^{-p(t-v)}dv$

then: $\boxed{C(t) = (p+\theta)[W(t)+H(t)]}$

5. *Dynamic Equations*

To get the dynamic equations we use the Leibniz rule for differentiation:

$\left(\frac{d}{dt}\int_{a(t)}^{b(t)} f(x,t)dx\right) = \int_{a(t)}^{b(t)} \frac{\partial f}{\partial t}dx + f(b(t),t)\dot{b}'(t) - f(a(t),t)\dot{a}'(t)$

Wealth:

$\dot{W}(t) = \cancelto{0}{pw(t,t)} - \int pw(vt)(-p)e^{-p(t-v)}dv + \int_{-\infty}^{t} w_t(v,t)pe^{-p(t-v)}dv$
$\dot{W}(t) = -pW(t) + \int_{-\infty}^{t} w_t(v,t)pe^{-p(t-v)}dv$

replace $w_t(v,t) = y(v,t) + (r(t)+p)w(v,t) - c(v,t)$ from the BC:

$\dot{W}(t) = \int_{-\infty}^{t}[y(v,t)-c(v,t)]pe^{-p(t-v)}dv + (r(t)+\cancel{p}-\cancel{p})W(t)$
$\dot{W}(t) = \int_{-\infty}^{t}[y(v,t)-c(v,t)]pe^{-p(t-v)}dv + r(t)W(t)$

$$\boxed{\dot{W}(t) = Y(t) + r(t)W(t) - C(t)}$$

Income:

$H(t) = \int_{-\infty}^{t} h(v,t)pe^{-p(t-v)}dv$

$H(t) = \int_{-\infty}^{t}\left(\int_{t}^{\infty} y(v,\tau)e^{-\int_t^\tau [r(s)+p]ds}d\tau\right)pe^{-p(t-v)}dv$

Change the order of integration,

$$H(t) = \int_{t}^{\infty}\underbrace{\left(\int_{-\infty}^{t} y(v,\tau)pe^{-p(t-v)}dv\right)}_{Y(\tau)}e^{-\int_t^\tau [r(s)+p]ds}d\tau$$

Assume $y(v,t) = Y(t) \quad \forall t$ then $Y(\tau)\cancelto{1}{\int_{-\infty}^{t} pe^{-p(t-v)}} = Y(\tau)$

$H(t) = \int_{t}^{\infty} Y(\tau)e^{-\int_t^\tau [r(s)+p]ds}d\tau$

Now using the Leibniz rule:

for the interior derivative in the following function notice that:
$\frac{d}{dt}\left(-\int_t^\tau (r(s)+p)ds\right) = \frac{d}{dt}\left(-(R(\tau)-R(t)) - P(\tau - t)\right) = r(t)+p$

$$\dot{H}(t) = 0 - Y(t)e^{\cancelto{0}{-\int_t^t (r(s)+p)ds}} + (r(t)+p)\int_t^\infty Y(\tau)e^{-\int_t^\tau [r(s)+p]ds}d\tau$$

$$\boxed{\dot{H}(t) = (r(t)+p)H(t) - Y(t)}$$

finally,

$$\boxed{\dot{C}(t) = (p+\theta)[\dot{W}(t) + \dot{H}(t)]}$$

subs. $\dot{W}(t), \dot{H}(t)$:

$$\begin{aligned} \cdot C(t) &= (p+\theta)(\cancel{Y(t)} + r(t)W(t) - C(t) + (r(t)+p)H(t) - \cancel{Y(t)}) \\ &= (p+\theta)(r(t)W(t) - C(t)) + (r(t)+p)\underbrace{[C(t) - (p+\theta)W(t)]}_{(p+\theta)H(t)} \end{aligned}$$

$$\boxed{\dot{C}(t) = (r(t)-\theta)C(t) - p(p+\theta)W(t)}$$

Example: Small Open Economy dynamics

- $r(t) = r$ (small economy)
- Assets: $F = W$
- Non asset income: $Y(t) = w$

$$\dot{C}(t) = (r+\theta)C(t) - p(p+\theta)F(t)$$
$$\dot{F}(t) = w + r \cdot F(t) - C(t)$$

Slope of $\dot{C} = 0$ line:

$\frac{dC}{dF} = \frac{p(p+\theta)}{(r-\theta)}$

Slope of $\dot{F} = 0$ line:

$\frac{dC}{dF} = r$

11 Growth Models

11.1 Harrod-Domar

$Y = vK$
$I = S = \Delta K$
$S = sY$

$\Rightarrow \quad \Delta Y = v\Delta K = v \cdot I = v \cdot sY \Rightarrow \quad \mathbf{\frac{\Delta Y}{Y} = vs}$

then output rate is the output to capital ratio times the savings rate.

11.2 Solow-Swan

- Exogenous savings rate
- Model with depreciation δ and population growth n

$Y = F(K, L)$ (F is assumed to be HOD1)
$S = sF(K, L)$
$\dot{K} = I - \delta K$
$S = I$

$\frac{\dot{L}}{L} = n$

put the dynamic equation in per-capita terms:

$\frac{\dot{K}}{L} = \frac{sF(K,L)}{L} - \delta\frac{K}{L}$
$\dot{k} + nk = sf(k) - \delta k$

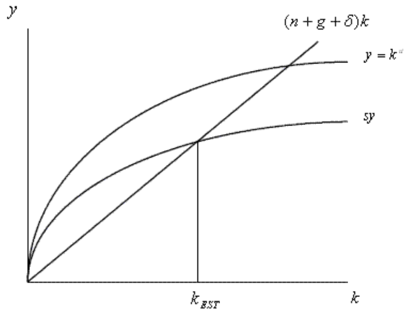
$$\boxed{\dot{k} = sf(k) - (\delta + n)k}$$

Linearize the equation:

$\dot{k} = (sf'(k) - (\delta + n))[k - \tilde{k}]$

at the SS:

$sf'(k) = \delta + n$



other than in k^* the depreciation exceeds the level of investment of the opposite, making k^* the only level of percapita capital that is steady.

in the SS all variable grow at the same rate:

$\frac{\dot{K}}{K} = n$:

$0 = \dot{k} = \frac{\dot{K}}{L} - nk$
$\frac{1}{K}\frac{\dot{K}}{L} = n\frac{K}{L}\frac{1}{K} \quad \Rightarrow \quad \frac{\dot{K}}{K} = n$

$\frac{\dot{Y}}{Y} = n$: by HOD1 output grows at the rate that factors grow times their shares. Since both L, K grow at n then output does it as well.

finding the max. level of consumption:

$$c(k^*) = f(k^*) - \underbrace{(\delta + n)k^*}_{sf(k)}$$

$c'(k^*) = f'(k^*) - (\delta + n) = 0 \quad \Rightarrow \quad \boxed{f'(k^*) = (\delta + n)}$ Golden rule: Level of k that maximizes c.

such k^* is denoted as the dynamically efficient capital level.

Savings rate (optimal):

$$
\begin{aligned}
c &= \overbrace{\cancel{f'\cdot k} + \overbrace{(f - f'\cdot k)}^{w}}^{f} - \cancel{(n+\delta)k} \\
&= f - f'\cdot k = w
\end{aligned}
$$

$$
\begin{aligned}
&f - c = f'\cdot k \\
&sf(k^*) = f'(k^*)k^* \quad \Rightarrow \quad \boxed{s = \tfrac{f'(k^*)k^*}{f(k^*)}}
\end{aligned}
$$

since $f'(k^*)k = (n+\delta)k$ the optimal savings rate is set such that just compensates for depreciation and aging (pop. growth).

11.3 Ramsey

- First growth model based on intertemporal optimization (microfounded).

HH UMP:

Budget Constraint:

$\dot{K} = wL + rK - C$

in per-capita terms: $\frac{\dot{K}}{L} = w + rk - c$

rearrange in terms of $\dot{k}$ (since $\dot{k} = (\dot{\frac{K}{L}})$): $\dot{k} + nk = w + rk - c$

$$\boxed{\dot{k} = w + (r-n)k - c}$$

$$H = \ln ce^{-\beta t} + \lambda e^{-\beta t}[w + (r+n)k - c - \dot{k}]$$

FOC:

$[c]: \qquad \frac{1}{c} = \lambda$

$[k]: \qquad \lambda(r-n) = -(\dot{\lambda} - \beta\lambda) \quad \Rightarrow \quad (r-n) = \beta - \frac{\dot{\lambda}}{\lambda}$

Firm PMP:

$\pi = f(k) - (\mathbf{r} + \delta)k - w$ Notice: Firms discount depreciation (δ) from their profits.

$[k]: \qquad f'(k) = r + \delta \quad \Rightarrow \quad w = f(k) - f'(k)\cdot k$

Dynamic equations

subs. $r = f'(k) - \delta$ in the second FOC:

$$f'(k) - \delta - n = \beta - \frac{\dot{\lambda}}{\lambda} \quad \Rightarrow \quad -\frac{\dot{\lambda}}{\lambda} = f'(k) - \delta - n - \beta$$

Consumption growth:

$$
\begin{aligned}
&(\dot{\frac{1}{c}}) = \dot{\lambda} \\
&-\frac{1}{c^2}\dot{c} = \dot{\lambda} \quad \Rightarrow \quad \frac{\dot{c}}{c} = -\frac{\dot{\lambda}}{\lambda}
\end{aligned}
$$

$\frac{\dot{c}}{c} = f'(k) - (\delta + n + \beta)$

$$\boxed{\dot{c} = c[f'(k) - (\delta + n + \beta)]}$$

Capital:

Replace $w = f - f'k$ and $\mathbf{r} = \mathbf{f}' - \boldsymbol{\delta}$ in the HH BC:

$\dot{k} = w + (r - n)k - c$

$$\boxed{\dot{k} = f(k) - (n + \delta)k - c}$$

In SS:

$$\begin{aligned} &\dot{k} = 0 \quad \Rightarrow \quad f(k) = (\delta + n)k + c \\ &\dot{c} = 0 \quad \Rightarrow \quad \boxed{f'(k) = n + \delta + \beta} \end{aligned}$$

Ramsey capital at the SS, i.e., dynamic inefficiency outcome. The resulting capital is lower than optimal.

11.4 Romer

- There are capital (K) externalities externalities, i.e., ceteris paribus a country (even in per-capita terms) is better the more capital they have (**scale effect**).
- Given this setup the model is known as AK-model
- This model doesn't have population growth

$$\begin{aligned} y &= Ak^{\alpha}K^{1-\alpha} \\ &= Ak^{\alpha}k^{1-\alpha}N^{1-\alpha} \\ &= AN^{1-\alpha}k \end{aligned}$$

The scale effects are seen here as well (are a consequence of externalities in aggregate K) but given the production function it implies the more population the greater the per capita output.

Since there is no population growth notice that the per-capita output grows at the same rate that the per-capita capital,

$\dot{y} = AN^{1-\alpha}\dot{k} \quad \Rightarrow \quad \frac{\dot{y}}{y} = \frac{\dot{k}}{k}$

HH UMP:

BC: taxes are included (both proportional and lump sum)

$$\boxed{(1 - \tau)Ak^{\alpha}K^{1-\alpha} - c - T - \dot{k} = 0}$$

$$H = \ln ce^{-\beta t} + \lambda e^{-\beta t}[(1 - \tau)Ak^{\alpha}K^{1-\alpha} - c - T - \dot{k}]$$

FOC: $(H_c = 0, \quad H_k = -\dot{\mu})$

$[c]: \quad \frac{1}{c} = \lambda$

$[k]: \quad (1-\tau)A\alpha k^{\alpha-1}K^{1-\alpha} = \beta - \frac{\dot{\lambda}}{\lambda} \quad \rightarrow \quad (1-\tau)A\alpha N^{1-\alpha} = \beta - \frac{\dot{\lambda}}{\lambda}$

given that $\frac{\dot{c}}{c} = -\frac{\dot{\lambda}}{\lambda} \quad \Rightarrow \quad \frac{\dot{c}}{c} = (1-\tau)A\alpha N^{1-\alpha} = \phi$

$\frac{\dot{c}}{c} = \phi$ (consumption grows at a constant rate)

Consumption function

$\dot{c} = c\phi$
$\dot{c} - \phi c = 0$
$\frac{d}{dt}\left(ce^{-\phi t}\right) = 0$
$ce^{-\phi t} = c_0 \quad \Rightarrow \quad \boxed{c(t) = c_0 e^{\phi t}}$

Capital

HH BC: $\dot{k} = (1-\tau)AN^{1-\alpha}k - c - T$
Gov BC: $\tau AN^{1-\alpha}k = -T$

Summing the BC we get the global feasibility constraint:

$\dot{k} = AN^{1-\alpha}k - c$

subs. $c(t)$,

$\dot{k} = AN^{1-\alpha}k - c_0 e^{\phi t}$

let $AN^{1-\alpha} = \theta$,

$$\dot{k} - \theta k = -c_0 e^{\phi t}$$

$$(\dot{k} - \theta k)e^{-\theta t} = -c_0 e^{\phi t} e^{-\theta t}$$

$$\frac{d}{dt}\left(ke^{-\theta t}\right) = -c_0 e^{\phi t} e^{-\theta t}$$

$$ke^{-\theta t} = k_0 - \int_0^t c_0 e^{(\phi-\theta)s} ds$$

by applying the TVC we know that $k_0 = \int_0^\infty c_0 e^{(\phi-\theta)s} ds$ then,

$ke^{-\theta} = \int_t^\infty c_0 e^{(\phi-\theta)s} ds$

$ke^{-\theta} = c_o \left[\frac{e^{(\phi-\theta)s}}{\phi-\theta}\right]_t^\infty$

$ke^{-\theta t} = -\frac{c_0}{\phi-\theta} e^{(\phi-\theta)t}$

$\boxed{k(t) = \frac{c_0}{\theta-\phi} e^{\phi t}}$

additionally,

$$\dot{k} = \phi \tfrac{c_0}{\theta-\phi} e^{\theta t} = \phi k \quad \Rightarrow \quad \dot{k} = \phi k$$

$$\tfrac{\dot{k}}{k} = \phi \quad \Rightarrow \quad \tfrac{\dot{y}}{y} = \phi$$

11.5 Comparison between growth models:

The golden rule states that: $f'(k) = n + \delta$

or in the Ramsey model (where the firm discount the depreciation from their profits) that: $\overbrace{f'(k) - \delta}^{r} = n$, this can be tested in a basic OLG setup.

$$U = \ln c_t + \tfrac{1}{1+\theta} \ln c_{t+1} \quad \text{s.t.} \quad c_t + \tfrac{c_{t+1}}{1+r_{t+1}} = w_t \quad \text{(Lifetime BC)}$$

FOC: $[c_t] : \quad \frac{1}{c_t} = \lambda$

$[c_t + 1] : \quad \frac{1}{1+\theta}\frac{1}{c_t} = \frac{1}{1+r_{t+1}}\lambda$

$$\Rightarrow \quad c_{t+1} = \left(\tfrac{1+r_{t+1}}{1+\theta}\right) c_t$$

subs in the definition of c_{t+1}:

$$c_{t+1} = (1 + r_{t+1})s_t = \left(\tfrac{1+r_{t+1}}{1+\theta}\right) c_t \quad \Rightarrow \quad c_t = (1+\theta)$$

subs c_t in the BC for the first period:

$$c_t + s_t = w_t \quad \Rightarrow \quad (2+\theta)s_t = w_t$$

$$\boxed{s_t = \tfrac{w_t}{2+\theta} = \tfrac{(1-\alpha)Ak^\alpha}{2+\theta}} \qquad \text{where } w_t = f(k) - f'(k) \cdot k$$

Also, assume we are about the SS so that $k_t = k_{t+1}$ and therefore:

$$(1+n)k_t = s_t$$

$$(1+n)k_t = \tfrac{(1-\alpha)Ak^\alpha}{2+\theta} \quad \Rightarrow \quad \tfrac{(1+n)(2+\theta)}{(1-\alpha)} = Ak_t^{\alpha-1}$$

Now let us consider:

$f(k) = Ak^\alpha - \delta k$ (net output given depreciation - Ramsey prod. f/n.).

Notice this function is consistent with the definition of w_t above. It is important to mention that the depreciation of k can be taken out from the output function directly or from the return of capital (as long as it is only taken out once), in both cases it must hold that $r = \alpha Ak^{\alpha-1} - \delta$

$$r = \alpha Ak^{\alpha-1} - \delta$$

$$r = \tfrac{(1-\alpha)Ak^\alpha}{2+\theta} \quad \Rightarrow \quad \boxed{\tfrac{(1+n)(2+\theta)}{(1-\alpha)} - \delta \gtreqless n}$$

compared with the golden rule ($r - \delta = n$) we obtain that the OLG model can be dinamically inefficient.

Golden rule in Ramsey model

in Solow model we have: $\dot{k} = sf(k) - (n + \delta)k$ and in Ramsey model the same statement is obtained: $\dot{k} = f(k) - c - (n + \delta)k$.

Therefore the Golden rule is be the same (i.e. the level of capital that maximizes the consumption is the same in both models): k^* is such that $f(k^*) = n + \delta$

However in the Ramsey model the consumption growth is given by:

$\frac{\dot{c}}{c} = f'(k) - (n + \delta + \beta)$ and then in SS the level of capital is such that: $\boxed{f'(k^{ss}) = n + \delta + \beta}$.

Given that $f_{kk} < 0$ and $f'(k^*) < f'(k^{ss})$ we get that $k^* > k^{ss}$, i.e., the level of capital obtained in the SS of the Ramsey model implies a dynamic inefficient result (the optimal accumulated capital should be greater).

Comparison between Romer and Harrod-Domar

- H-D is a labor neutral model (only capital implied) whereas Romer has both factors but externalities on aggregate capital. Such externalities leads to the AK representation, i.e., in percapita terms the effects of labor are not present and we are just left with a scale effect.

- These features make the models similar and we will see how in the Romer model we also get that the rate of change of capital depends on the savings rate times the ratio capital to output: $\frac{\dot{k}}{k} = s \cdot v$

In H-D $\frac{\dot{Y}}{Y} = sv$:

$$Y = vK \quad \rightarrow \quad \dot{Y} = v\dot{K} = vsY \Rightarrow \quad \frac{\dot{Y}}{Y} = v \cdot s$$

whereas in the Romer model:

$$y = AN^{1-\alpha}k$$
$$\phi = A\alpha N^{1-\alpha} - \beta$$

$$k = \frac{c(0)e^{\phi t}}{\phi - \theta}$$
$$\dot{k} = \phi k$$

$$y = AN^{1-\alpha}\,\overbrace{\frac{c(0)e^{\phi t}}{\phi - \theta}}^{k}$$

$$\frac{\dot{k}}{y} = \frac{\phi k}{AN^{1-\alpha}k} = \frac{\phi}{AN^{1-\alpha}} = \mathbf{s}$$

On the other hand,

$$\frac{Y}{K} = \frac{y}{k} = \frac{AN^{1-\alpha}k}{k} = AN^{1-\alpha} = \mathbf{v}$$

then: $\frac{\dot{k}}{k} = \frac{\dot{k}}{y}\frac{y}{k} = s \cdot v$

Notice that the results are analogous since the Romer model doesn't have population growth so that: $\frac{\dot{K}}{K} = \frac{\dot{k}}{k}$

Other comparisons

H-D: $\frac{\dot{K}}{K} = \frac{\dot{Y}}{Y} - sv \propto s$ the savings rate is constant through time and exogenous so there are no transition dynamics and no population growht in this model.

Solow: $\frac{\dot{k}}{k} = \frac{\dot{y}}{y} = \frac{\dot{L}}{L} = n$ (exogenous)

The level of capital in the Solow model is the one implied by the golden rule: $f'(k) = n + \beta$. Such level of capital is greater than in the Ramsey model ($k^{golden} > k^{ss,ramsey}$) and can be greater than the one implied in the OLG model (depends on the parameters).

Extensions to the growth models

Several extensions change the form of the production function: e.g. they can include other factors $F(K, L, t)$

$Y = T(t)F(K, L)$	Hicks neutral (ratios of factors don't change)
$Y = F(K, A(t)L)$	Harrod neutral (labor augmented)
$Y = F(B(t)K, L)$	K augmenting

Part III

Workhorses for the RBC model

12 Arrow-Debreu Economy

A competitive Arrow-Debreu (A-D) equilibrium are prices $\{\hat{p}_t\}_{t=0}^{\infty}$ and allocations $\left(\{\hat{c}_t^i\}_{t=0}^{\infty}\right)_{i=1,2}$ such that:

1. Given $\hat{\mathbf{p}}$, for all i, $\{\hat{c}_t^i\}_{t=0}^{\infty}$ solves:

$$\max_{\{c_t^i\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t \ln(c_t^i)$$

s.t.

$$\sum_{t=0}^{\infty} \hat{p}_t c_t^i \leq \sum_{t=0}^{\infty} \hat{p}_t e_t^i$$

$$c_t^i \geq 0 \qquad \forall t$$

2.

$$\hat{c}_t^1 + \hat{c}_t^2 = e_t^1 + e_t^2 \qquad \forall t$$

Solving for the equilibrium:

assume the deterministic endowment stream given by:

$$e_t^1 = \begin{cases} 2 & \text{if } t \text{ is even} \\ 0 & \text{if } t \text{ is odd} \end{cases}, \; e_t^2 = \begin{cases} 0 & \text{if } t \text{ is even} \\ 2 & \text{if } t \text{ is odd} \end{cases}$$

$$L = \sum_t \beta^t \ln c_t^i + \lambda_i \left(\sum_t p_t (e_t^i - c_t^i) \right)$$

$$[c_t^i] : \qquad \frac{\beta^t}{c_t^i} = \lambda_i p_t$$
$$[c_{t+1}^i] : \qquad \frac{\beta^{t+1}}{c_{t+1}^i} = \lambda_i p_{t+1}$$

then: $p_{t+1} c_{t+1}^i = \beta p_t c_t^i$

sum over i: $\quad p_{t+1} \underbrace{\left(c_{t+1}^1 + c_{t+1}^2 \right)}_{e_{t+1}^1 + e_{t+1}^2 = 2} = \beta p_t \underbrace{\left(c_t^1 + c_t^2 \right)}_{e_t^1 + e_t^2 = 2}$

hence: $\quad p_{t+1} = \beta p_t$
in general: $\quad p_t = \beta^t p_0$

Normalize $p_0 = 1 \quad \Rightarrow \quad \hat{p}_t = \beta^t$,
subs. in $p_{t+1} c_{t+1}^i = \beta p_t c_t^i \quad \rightarrow \quad \beta^{t+1} c_{t+1}^i = \beta \beta^t c_t^i$

we get that $\boxed{c_{t+1}^i = c_t^i}$

subs. in the LHS of the BC: $\sum_{t=0}^{\infty} \hat{p}_t c_t^i = c_0^i \sum_{t=0}^{\infty} \beta^t$
$= c_0^i \frac{1}{1-\beta}$

Now consider the RHS:

for $i = 1$,

$$\begin{aligned} \sum \hat{e}_t^1 &= (\beta^0 2 + \beta^1 0 + \beta^2 2 + \beta^3 0 + \cdots) \\ &= 2(1 + \beta^2 + \beta^4 + \cdots) \\ &= 2 \sum \beta^{2t} = \frac{2}{1-\beta^2} \end{aligned}$$

for $i = 2$,

$$\begin{aligned} \sum \hat{e}_t^2 &= (\beta^0 0 + \beta^1 2 + \beta^2 0 + \beta^3 2 + \cdots) \\ &= 2\beta(1 + \beta^2 + \beta^4 + \cdots) \\ &= 2\beta \sum \beta^{2t} = \frac{2\beta}{1-\beta^2} \end{aligned}$$

therefore: LHS = RHS

$$c_t^1 = c_0^1 = (1-\beta)\frac{2}{1-\beta^2} = \frac{2}{1+\beta} > 1$$

$$c_t^2 = c_0^2 = (1-\beta)\frac{2\beta}{1-\beta^2} = \frac{2\beta}{1+\beta} < 1$$

then in the equilibrium the first agent is better off due to the fact that he receives positive endowments sooner, when they are better regarded. Additionally trade is the best option because in autharky the logaritmic function implies a lifetime utility of $u(e^i) = -\infty$

Pareto Optimal Allocation and First Welfare Theorem:

Pareto Optimal allocation: $\hat{c}^i$ is P.O. if feasible ($\sum c_t^i \leq \sum e_t^i \quad \forall t, i$) and $\nexists \tilde{c}^i$ that is feasible and $u(\tilde{c}^i) \geq u(\hat{c}^i) \quad \forall i$ with at least one strict inequality for some i.

First Welfare Theorem: Let $\hat{c}^i$ be a W.E. then it is also a P.O.

Proof: (by contradiction)

Suppose not, then $\exists$ a $\tilde{c}_i$ that is feasible and: $u(\tilde{c}^i) \geq u(\hat{c}^i) \quad \forall i$ with some strict inequality. WLOG suppose the strict inequality holds for $i = 1$.

we have that necessarily: $\sum \hat{p}_t \tilde{c}_t^1 > \sum \hat{p}_t \hat{c}_t^1$ $\qquad$ if not then a better allocation could be bought
also: $\qquad \sum \hat{p}_t \tilde{c}_t^2 \geq \sum \hat{p}_t \hat{c}_t^2$

If not, i.e. $\sum \hat{p}_t \tilde{c}_t^2 < \sum \hat{p}_t \hat{c}_t^2$ then $\exists \delta > 0$ s.t. $\sum \hat{p}_t \tilde{c}_t^2 + \delta = \sum \hat{p}_t \hat{c}_t^2$
then let $\check{c}_t^2 = \tilde{c}_0^2 + \delta$ for $t = 0$ and $\check{c}_t^2 = \tilde{c}_t^2$ for $t \geq 1 \quad \Rightarrow \quad u(\check{c}^2) > u(\tilde{c}^2) \geq u(\hat{c}^2)$ which is contradictory since $\hat{c}^2$ is an equilibrium allocation.

then summing we have: $\sum_t \hat{p}_t (\tilde{c}_t^1 + \tilde{c}_t^2) > \sum_t \hat{p}_t (\hat{c}_t^1 + \hat{c}_t^2)$

Also by feasibility: $\tilde{c}_t^1 + \tilde{c}_t^2 = \hat{c}_t^1 + \hat{c}_t^2 = e_t^1 + e_t^2$ then:

$$\sum_t \hat{p}_t(e_t^1 + e_t^2) > \sum_t \hat{p}_t(e_t^1 + e_t^2) \qquad \rightarrow\leftarrow \blacksquare$$

Negishi's method: Method to compute equilibria for economies in which the welfare theorems hold, the idea is to get P.O. allocations solving a social planner's problem, i.e., with no prices involved.

Social Planner's Problem (SPP):

$$\max_{\{c_t^1, c_t^2\}_{t=0}^{\infty}} \alpha^1 u(c^1) + \alpha^2 u(c^2) = \sum_t \beta^t \left[\alpha^1 \ln c_t^1 + \alpha^2 \ln c_t^2\right]$$

s.t.

$$c_t^i \geq 0$$

$$c_t^1 + c_t^2 = e_t^1 + e_t^2 \equiv 2 \quad \forall t$$

Proposition: Any c^1, c^2 that solves the SPP for some $\alpha > 0$ is P.O.
Proposition: Any P.O. allocation c^1, c^2 is the solution to the SPP for some $\alpha \geq 0, \alpha \neq 0$

Solving the problem above:

$L = \sum_t \beta^t \left[\alpha^1 \ln c_t^1 + \alpha^2 \ln c_t^2\right] + \sum_t \frac{\mu_t}{2}(2 - c_t^1 - c_t^2)$

$[c_t^1] \qquad \frac{\alpha^1 \beta^t}{c_t^1} = \frac{\mu_t}{2}$
$[c_t^2] \qquad \frac{\alpha^2 \beta^t}{c_t^2} = \frac{\mu_t}{2}$

$\frac{c_t^1}{c_t^2} = \frac{\alpha^1}{\alpha^2} \quad \rightarrow \quad c_t^1 = \frac{\alpha^1}{\alpha^2} c_t^2,$

subs. in the BC: $\quad c_t^1 + c_t^2 = 2$
$\frac{\alpha^1}{\alpha^2} c_t^2 + c_t^2 = 2$

$\Rightarrow \quad c_t^2 = \frac{2}{1+\frac{\alpha^1}{\alpha^2}} = c_t^2(\alpha), \qquad c_t^1(\alpha) = \frac{2}{1+\frac{\alpha_2}{\alpha_1}}$

Summing FOCs (1) and (2):

$\mu_t = (\alpha^1 + \alpha^2)\beta^t \quad$ or $\quad \mu_t = \beta^t \quad$ with $\sum_i \alpha^i = 1$

then: PO: $(c^1, c^2) : \left\{ c_t^2 = \frac{2}{1+\frac{\alpha^1}{\alpha^2}} = c_t^2(\alpha), \quad c_t^1(\alpha) = \frac{2}{1+\frac{\alpha_2}{\alpha_1}} \text{ for } \frac{\alpha_1}{\alpha_2} \in [0, \infty) \right\}$

also notice that if pareto weights α^i and the Lagrange multipliers λ_i are related by $\lambda_i = \frac{1}{2\alpha^i}$ the FOC of both SPP and UMP are the same.

before we saw that in the AD equilibrium the prices were $\hat{p}_t = \beta^t$, i.e., $p_t = \mu_t$. It remains to show that the allocation found is feasible, to do so let us define a function of transfers for a given α:

$t^i(\alpha) = \sum_t \mu_t[c_t^i(\alpha) - e_t^i] \qquad$ (transfer function)

we look for α s.t. $t^i(\alpha) = 0$

the transfers are the amount of numeraire good that the agent needs to be transferred to be able to afford the P.O. allocation given α

subs. the condition $\mu_t = (\alpha^1 + \alpha^2)\beta^t$: $\quad t^i(\alpha) = \sum_t (\alpha^1 + \alpha^2)\beta^t [c_t^i(\alpha) - e_t^i]$

we can sum both transfer functions, they add to zero and can be expressed in terms of the resource constraint:

$\sum_{i=1}^2 t^i(\alpha) = \sum_{i=1}^2 \sum_t \mu_t [c_t^i(\alpha) - e_t^i] = \sum_{i=1}^2 \mu_t \underbrace{\sum_t [c_t^i(\alpha) - e_t^i]}_{0} = 0$ with this we have effectively one less equation, e.g. $(t^1(\alpha) = 0)$ and one unknown α^1/α^2

then reconsider normalization $(\alpha^1 + \alpha^2 = 1)$:

$t^1(\alpha) = \sum_t \beta^t [c_t^1(\alpha) - e_t^1]$

subs. $c_t^1(\alpha) = \frac{2}{1+\alpha_2/\alpha_1}$,

$t^1(\alpha) = \sum_t \beta^t [\frac{2}{1+\alpha_2/\alpha_1} - e_t^1] = \frac{2}{(1-\beta)(1+\alpha^2/\alpha^1)} - \frac{2}{\underbrace{1-\beta^2}_{(1-\beta)(1+\beta)}} = 0$

rearrange:

$\frac{1}{1+\alpha^2/\alpha^1} = \frac{1}{1+\beta} \quad \Rightarrow \quad \boxed{\frac{\alpha^2}{\alpha_1} = \beta}$

then: $c_t^1 = \frac{2}{1+\beta}$ and $c_t^2 = \frac{2\beta}{1+\beta}$
with $p_t = \mu_t = \beta^t$

Summary: (*Negishi's method*)

1. Solve SPP for the P.O. allocations with $\alpha = (\alpha^1, \alpha^2) > 0$
2. Compute transfers $t^i(\alpha)$ that make the allocation affordable. As prices use the lagrange multipliers.
3. Find normalized pareto relative weights that makes transfers be zero $(\hat{\alpha})$.
4. P.O. are the equilibrium allocations after substituting α. The supporting prices are the lagrange multipliers μ_t.

Note: to solve for the equilibrium directly involves solving an infinite number of equations in an infinite number of unknowns. The Neguishi methods reduces the computation to a finite number of equations and unknows in the step 3.

13 Sequential Markets

- Trade has place each period (not all predetermined at period 0 as before).

- Change is carried out using bonds. The bond is a claim to pay 1 unit of consumption good in period $t+1$ in exchange for $\frac{1}{1+r_{t+1}}$ units of the good in period t.

Sequential Markets equilibrium (SM)

SM equilibrium is allocations $\{(\hat{c}^i_t, \hat{a}^i_{t+1})_{i=1,2}\}^\infty_{t=0}$ and interest rates $\{\hat{r}_{t+1}\}^\infty_{t=0}$ such that for all i:

1. given interest rates $\{\hat{r}_{t+1}\}^\infty_{t=0}$, the allocation $\{(\hat{c}^i_t, \hat{a}^i_{t+1})_{i=1,2}\}^\infty_{t=0}$ solves:

$$\max \sum_{t=0}^{\infty} \beta^t \ln c^i_t$$

s.t.

$$c^i_t + \frac{a^i_{t+1}}{1+r_{t+1}} \leq e^i_t + a^i_t$$

$$c^i_t \geq 0 \quad \forall i, t$$

$$a^i_{t+1} \geq -A^{-i} \qquad \text{(Borrowing constraint)}$$

2. MC: for all $t \geq 0$:

$$\sum_{i=1}^{2} \hat{c}^i_t = \sum_{i=1}^{2} e^i_t$$

$$\sum_{i=1}^{2} a_{t+1^i} = 0$$

Proposition: *An allocation forming an A-D equilibrium can be supported as a SM equilibrium*

Let $\{(\hat{c}^i_t)_{i=1,2}\}^\infty_{t=0}, \{\hat{p}_t\}^\infty_{t=0}$ form an equilibrium with $\frac{\hat{p}_{t+1}}{\hat{p}_t} \leq \varepsilon < 1 \quad \forall t$ then $\exists (\bar{A}^i)_{i=1,2}$ and a SM equilibrium $\{(\tilde{c}^i_t, \tilde{a}^i_{t+1})\}^\infty_{t=0}$ and $\{r_{t+1}\}^\infty_{t=0}$ s.t. $\mathbf{\tilde{c}^i_t} = \mathbf{\hat{c}^i_t} \quad \forall i, t.$

Reversely, let $(\hat{c}^i_t, \hat{a}^i_{t+1})^\infty_{t=0}, \{\hat{r}_{t+1}\}^\infty_{t=0}$ be a SM equilibrium. Suppose it satisfies $\hat{a}^i_{t+1} > -\bar{A}^i$ and $\hat{r}_{t+1} \geq \varepsilon > 0 \quad \forall i, t \quad \Rightarrow \exists$ and A-D equilibrium $\tilde{c}^i, \tilde{p}$ such that $\hat{c}^i_t = \tilde{c}^i_t \quad \forall t.$

Sketch of the proof:

1. Show that $BC_{AD} = BC_{SM}$
2. Find $\bar{A}^i$ large enough such that AD solution doesn't violate the no Ponzi condition when made a SM equilibrium.
3. Show SM equilibrium can be made into an AD equilibrium. This will be given because the additional constraint $\hat{a}^i_{t+1} \geq -\bar{A}^i$ is not binding, then even if the set over which the maximization is done is bigger in the AD framework, the additional constraint is not conflicting and the optimal would be the same in both cases.

The most important part is the step 1, the rest can be argued more briefly.

Step 1: $BC_{AD} = BC_{SM}$

Set $\hat{p}_0 = 1$

use $1 + \hat{r}_{t+1} = \frac{\hat{p}_t}{\hat{p}_{t+1}}$

and consider the sequence of SM BC:

$c^i_0 + \frac{a^i_1}{1+\hat{r}_1} = e^i_0$
$c^i_1 + \frac{a^i_2}{1+r_2} = e^i_1 + a^i_1$

$\vdots$

$c^i_t + \frac{a^i_{t+1}}{1+\hat{r}_{t+1}} = e^i_t + a^i_t$

subs. for a^i_1 in first BC from the second one:

$c^i_0 + \frac{c^i_1}{1+\hat{r}_1} + \frac{a^i_2}{(1+\hat{r}_1)(1+\hat{r}_2)} = e^i_0 + \frac{e^i_1}{1+\hat{r}_1}$

keep doing the same procedure to leave only the last bond present in the BC:

$\sum_{t=0}^{T} \frac{c_t^i}{\prod_{j=1}^{t}(1+\hat{r}_j)} + \frac{a_{T+1}^i}{\prod_{j=1}^{T+1}(1+\hat{r}_j)} = \sum_{t=0}^{T} \frac{e_t^i}{\prod_{j=1}^{t}(1+\hat{r}_j)}$

Note: $\prod_{j=1}^{t}(1+\hat{r}_j) = \frac{\hat{p}_0}{\hat{p}_1}\frac{\hat{p}_1}{\hat{p}_2}\frac{\hat{p}_2}{\hat{p}_3}\cdots\frac{\hat{p}_{t-1}}{\hat{p}_t} = \frac{p_0}{p_t} = \frac{1}{\hat{p}_t}$

subs. and take limits as $T \to \infty$,

$\sum_{t=0}^{\infty} \hat{p}_t c_t^i + \lim_{T\to\infty} \frac{a_{T+1}^i}{\prod_{j=1}^{T+1}(1+\hat{r}_j)} = \sum_{t=0}^{\infty} \hat{p}_t e_t^i$

also:
$\lim_{T\to\infty} \frac{a_{T+1}^i}{\prod_{j=1}^{T+1}(1+\hat{r}_j)} \geq 0$ (the expression is at the least zero)
$\lim_{T\to\infty} \prod_{j=1}^{T+1}(1+\hat{r}_j) = \infty$ (the denominator explodes)

then taking out the bond term we have:

$\boxed{\sum_{t=0}^{\infty} \hat{p}_t c_t^i \leq \sum_{t=0}^{\infty} \hat{p}_t e_t^i}$

14 New-classical Growth Model (NGM)

- 3 goods: labor n_t, capital k_t and final good ouput y_t
- y_t can be consumed c_t or invested i_t
- Technology: $y_t = F(k_t, n_t)$
  $y_t = c_t + i_t$
  $k_{t+1} = (1-\delta)k_t + i_t \quad \Rightarrow \quad i_t = k_{t+1} - (1-\delta)k_t$
- $k_{t+1} \geq 0$ but not necessarilly $i_t \geq 0$, i.e., capital can be disinvested and eaten.
- Information: No risk, HH and firms have perfect foresight
- Preferences: $u(c) = \sum_{t=0}^{\infty} \beta^t U(c_t)$

SPP - Sequential formulation:

$$w(\bar{k}_0) = \max_{c_t, k_t, n_t} \sum_{t=0}^{\infty} \beta^t U(c_t)$$

$$\begin{aligned} \text{s.t.} \quad F(k_t, n_t) &= c_t + k_{t+1} - (1-\delta)k_t \\ c_t \geq 0, k_t &\geq 0, 0 \leq n_t \leq 1 \\ k_0 &\leq \bar{k}_0 \end{aligned}$$

$U(\cdot) \subset C$ functions (continuously, differentiable), $F(\cdot)$ is HOD1 and both safisfy the inada conditions. Using these properties define:

$$f(k) = F(k,1) + (1-\delta)k \qquad \text{(amount of final good available for c or i)}$$

with f the consumption can be expressed as: $c_t = f(k_t) - k_{t+1}$

then SPP can be simplified to:

$$w(\bar{k}_0) = \max_{k_{t+1}} \sum_{t=0}^{\infty} \beta^t U(f(k_t) - k_{t+1})$$

$$\begin{aligned} \text{s.t.} \qquad 0 \leq k_{t+1} &\leq f(k_t) \\ k_0 &\leq \bar{k}_0 \text{ given} \end{aligned}$$

as stated before in the Dynamic Programming section this problem is equivalent to the following Bellman equation:

$$w(k_0) = \max_{0 \leq k_1 \leq f(k_0)} U(f(k_0) - k_1) + \beta w(k_1)$$

The solution methods for the former are the guess and verify method, the analytical method and the numerical approach. Refer to the DP section for further details.

Euler Equation and Transversality condition approach

In general the numerical approach will approximate the solution to the SPP problem. Notwithstanding, in some cases the sequential problem can be solved directly too. In such setup the finite and infinite horizon problems are similar except that the latter uses the TVC condition.

The SPP is:

$$\begin{aligned} w^T(\bar{k}_0) = & \max_{\{k_{t+1}\}_{t=0}^{T}} \sum_{t=0}^{T} \beta^t U(f(k_t) - k_{t+1}) \\ \text{s.t.} \qquad & 0 \leq k_{t+1} \leq f(k_t) \\ & k_0 \leq \bar{k}_0 \text{ given} \end{aligned}$$

$$L = U(f(k_0) - k_1) + \cdots + \beta^t U(f(k_t) - k_{t+1}) + \beta^{t+1} U(f(k_{t+1}) - k_{t+2}) + \cdots + \beta^T U(f(k_T) - k_{T+1})$$

FOC:

$$[k_{t+1}]: \qquad U'(f(k_t) - k_{t+1}) = \beta U'(f(k_{t+1}) - k_{t+2}) f'(k_{t+1}) \qquad \forall t = 0\bar{,}T \qquad \text{(Euler Equation)}$$

Ex: $U(c) = \ln(c)$, $f(k) = k^\alpha$ then the EE is:

$$\frac{1}{k_t^\alpha - k_{t+1}} = \beta \frac{1}{k_{t+1}^\alpha - k_{t+2}} \alpha k_{t+1}^{\alpha-1}$$

rerranging: $$k_{t+1}^\alpha - k_{t+2} = \alpha\beta d_{t+1}^{\alpha-1}(k_t^\alpha - k_{t+1})$$

define $z_t = \frac{k_{t+1}}{k_t^\alpha}$ (fraction of output saved). Divide by k_{t+1}:

$$\begin{aligned} 1 - z_{t+1} &= \frac{\alpha\beta(k_t^\alpha - k_{t+1})}{k_{t+1}} = \alpha\beta\left(\frac{1}{z_t} - 1\right) \\ z_t &= \frac{\alpha\beta}{1 + \alpha\beta - z_{t+1}} \end{aligned}$$

then:

$$\begin{aligned} k_{t+1} &= \alpha\beta \frac{1 - (\alpha\beta)^{T-t}}{1 - (\alpha\beta)^{T-t+1}} k_t^\alpha \\ c_t &= \frac{1 - \alpha\beta}{1 - (\alpha\beta)^{T-t+1}} k_t^\alpha \end{aligned}$$

for the infinite take logs of the variables $\ln c_t$, $\ln k_t$, replace that of capital in the former and replace in the value function.

In this specific case we can also find the limit as $T \to \infty$ of the k_{t+1} function:

$$\lim_{T\to\infty} k_{t+1} = \lim_{T\to\infty} \alpha\beta \frac{1-(\alpha\beta)^{T-t}}{1-(\alpha\beta)^{T-t+1}} k_t^{\alpha}$$
$$\lim_{T\to\infty} k_{t+1} = \alpha\beta k_t^{\alpha}$$

it is important to mention that is not the case in general that the optimal policy is the limit of the optimal policy for the T-horizon case.

Infinite horizon case

$$\begin{aligned} w^T(\bar{k}_0) &= \max_{\{k_{t+1}\}_{t=0}^{\infty}} \sum_{t=0}^{T} \beta^t U(f(k_t) - k_{t+1}) \\ \text{s.t.} \qquad & 0 \leq k_{t+1} \leq f(k_t) \\ & k_0 \leq \bar{k}_0 \text{ given} \end{aligned}$$

the period utility is strictly concave and the constraint set is convex, then the first order conditions are necessary conditions to characterize an optimal sequence $\{k_{t+1}\}_{t=0}^{\infty}$. The FOC w.r.t. k_{t+1} gives the Euler Equation:

$$U'(f(k_t) - k_{t+1}) = \beta U'(f(k_{t+1}) - k_{t+2}) f'(k_{t+1}) \qquad \forall t = 0, 1, \cdots, t, \cdots$$

instead of a terminal condition (absent because of the infinite nature of the problem) we use a transversality condition:

$$\lim_{t\to\infty} \beta^t U'(f(k_t) - k_{t+1}) f'(k_t) k_t = 0$$

The TVC says that the value of the capital stock measured in terms of discounted utility goes to zero as time goes to infinity.

TVC (standard definition)

the TVC stated above is $\lim_{t\to\infty} \beta^t U'(f(k_t) - k_{t+1}) f'(k_t) k_t = 0$ this is analog to the its common definition:

$$\lim_{t\to\infty} \lambda_t k_{t+1} = 0$$

where λ_t is the Lagrange multiplier on the ocnstraint $c_t + k_{t+1} = f(k_t)$.

The FOCs of the SPP state that:

$$\beta^t U'(c_t) = \lambda_t$$
$$\beta^t U'(f(k_t) - k_{t+1}) = \lambda_t$$

then the TVC is:

$$\lim_{t\to\infty} \beta^t U'(f(k_t) - k_{t+1}) = \lambda_t$$

this TVC is equivalent to the initially stated, to get from one to the other we can rearrange the expression and use the EE:

$$\lim_{t\to\infty} \beta^t U'(f(k_t) - k_{t+1})k_{t+1} = 0$$

$$\lim_{t\to\infty} \beta^{t-1} U'(f(k_{t-1}) - k_t)k_t = 0 \qquad \text{(in t-1)}$$

$$\lim_{t\to\infty} \beta^{t-1} \overbrace{\beta U'(f(k_t) - k_{t+1})f'(k_t)}^{\text{RHS of EE}} k_t = 0 \qquad \text{(using EE in t-1)}$$

$$\lim_{t\to\infty} \beta^t U'(f(k_t) - k_{t+1})f'(k_t)k_t = 0$$

[Theorem] *Sufficiency of the EE and TVC*

let $U(\cdot), F(\cdot), \beta$ (and thus $f(\cdot)$) satisfy the assumptions 1 and 2 (U,F is a C^1 function satisfying the inada conditions and F is HOD1) then an allocation $\{k_{t+1}\}_{t=0}^{\infty}$ that satisfies the Euler equations and the TVC solves the sequential social planners problem, for a given k_0.

with these result we can continue the example for the logaritmic utility and CD production function. The TVC is:

$$\lim_{t\to\infty} \beta^t U'(f(k_t) - k_{t+1})f'(k_t)k_t = \lim_{t\to\infty} \frac{\alpha\beta^t k_t^\alpha}{k_t^\alpha - k_{t+1}} = \lim_{t\to\infty} \frac{\alpha\beta^t}{1 - \frac{k_{t+1}}{k_t^\alpha}} = \lim_{t\to\infty} \frac{\alpha\beta^t}{1 - z_t}$$

also the EE stated that: $z_{t+1} = 1 + \alpha\beta - \frac{\alpha\beta}{z_t}$

Now, we can solve the EE guessing an initial value for z_0, it turns that only one gues doesn't violate the TVC or the non-negativity constraint on capital or consumption: $z_0 = \alpha\beta$

$$\lim_{t\to\infty} \frac{\alpha\beta^t}{1 - z_t} = \lim_{t\to\infty} \frac{\alpha\beta^t}{1 - \alpha\beta} = 0$$

therefore from the definition of z_t we have that $\boxed{k_{t+1} = \alpha\beta k_t^\alpha}$

Steady State:

in the SS $c_{t+1} = ct$ and the EE becomes:

$$\beta f'(k^*) = 1$$

$$f'(k^*) = 1 + \rho$$

and since $f'(k^*) = F_k(k, 1) + 1 - \delta$ then:

$$F_k(k^*, 1) - \delta = \rho \qquad \text{(modified golden rule)}$$

in the example:

$$\alpha k^{*\alpha-1} = \frac{1}{\beta} \quad \Rightarrow \quad k^* = (\alpha\beta)^{\frac{1}{1-\alpha}}$$

we can get the same result using the optimal policy function with $k_{t+1} = k_t = k^*$:

$$k = \alpha\beta k^{\alpha} \quad \Rightarrow \quad k^* = (\alpha\beta)^{\frac{1}{1-\alpha}}$$

finally the modified golden rule comes from the fact that in this case the social planner finds optimal to set the capital $k^* < k^{golden}$ as seen in previous sections (dynamically inefficient outcome). This means that the social planner considers the impatience of the representative agent.[7]

Remark on balanced growth

Consider populatin growth and labor augmenting technology: $N_t = (1+n)^t$

Labor augmenting technology: $F(K_t, (1+g)^t N_t)$

In such case we define the growth adjusted per capita variables as:

$\tilde{c}_t = \frac{c_t}{(1+g)^t}$
$\tilde{k}_t = \frac{k_t}{(1+g)^t} = \frac{K_t}{(1+n)^t(1+g)^t}$

with these transformations the model with growth and technology is no harder to analyze than the benchmark model. To do this we have to redefine the time discount factor, deflate all per-capita variables by technological progress, all aggregate variables in addition population growth, and pre-multiply effective capital tomorrow by $(1+n)(1+g)$.

Competitive equilibrium growth

A-D equilibrium

prices $\{p_t, w_t, r_t\}_{t=0}^{\infty}$, allocations for the firm $\{c_t, i_t, x_{t+1}, k_t^s, n_t^s\}$ and alocations for the HH $\{c_t, i_t, x_{t+1}, k_t^s, n_t^s\}$ such that:

1. Given prices $\{p_t, w_t, r_t\}_{t=0}^{\infty}$ the allocation of the reprsentative firm $\{c_t, i_t, x_{t+1}, k_t^s, n_t^s\}$ solves PMP:

$$\begin{aligned} \pi = \max_{\{y_t, k_t, n_t\}} &\sum_{t=0}^{\infty} p_t(y_t - r_t k_t - w_t n_t) \\ \text{s.t.} \qquad y_t &= F(k_t, n_t) \quad \forall t \\ y_t, k_t, n_t &\geq 0 \end{aligned}$$

2. Given prices $\{p_t, w_t, r_t\}_{t=0}^{\infty}$ the allocation of the HH solves UMP:

$$\begin{aligned} &\max_{\{c_t, i_t, x_{t+1}, k_t, n_t\}} \sum_{t=0}^{\infty} \beta^t U(c_t) \\ \text{s.t.} \qquad &\sum_{t=0}^{\infty} p_t(c_t + i_t) \leq \sum_{t=0}^{\infty} p_t(r_t k_t + w_t n_t) + \pi \\ &x_{t+1} = (1-\delta)x_t + i_t \\ &0 \leq n_t \leq 1, 0 \leq k_t \leq x_t \end{aligned}$$

3. MC

---

[7] Recall that the consumption is $c = f(k) - k$ and the capital that maximizes the consumption satisfies $f'(k^{golden}) = 1$ or $F_k(k^{golden,1}) - \delta = 0$

$$y_t = c_t + i_t \qquad \text{(Goods mkt)}$$
$$n_t^d = n_t^s \qquad \text{(Labor mkt)}$$
$$k_t^d = k_t^s \qquad \text{(Capital services mkt)}$$

in the equilibrium we will have:

$$k_t = k_t^d = k_t^s$$
$$n_t = n_t^d = n_t^s$$

Also from the FF optimization and given the production function $F(\cdot)$ is HOD1:

$$r_t = F_k(k_t, n_t)$$
$$w_t = F_n(k_t, n_t)$$

then by Euler's Theorem $F(k_t, n_t) = r_t k_t + w_t n_t \quad \Rightarrow \quad \pi = 0$

On the HH side we will have $n_t = 1$, $k_t = x_t$ and therefore $i_t = k_{t+1} - (1-\delta)k_t$ and hence,

$$F(k_t, 1) = c_t + \overbrace{k_{t+1} - (1-\delta)k_t}^{i_t}$$
$$f(k_t) = c_t + k_{t+1}$$

then we can rewrite the HH UMP as:

$$\max_{\{c_t, k_{t+1}\}} \sum_{t=0}^{\infty} \beta^t U(c_t)$$
$$\text{s.t.} \quad \sum_{t=0}^{\infty} p_t(c_t + k_{t+1} - (1-\delta)k_t) \leq \sum_{t=0}^{\infty} p_t(r_t k_t + w_t)$$
$$c_t, k_{t+1} \geq 0 \quad \forall t \geq 0$$
$$k_0 \quad \text{given}$$

FOC:

$[c_t]: \quad \beta^t U'(c_t) = \mu p_t$
$[c_{t+1}]: \quad \beta^{t+1} U'(c_{t+1}) = \mu p_{t+1}$
$[k_{t+1}]: \quad \mu p_t = \mu(1 - \delta + r_{t+1})p_{t+1}$

The corresponding EE is:

$$\frac{\beta U'(c_{t+1})}{U'(c_t)} = \frac{p_{t+1}}{p_t} = \frac{1}{1 + r_{t+1} - \delta}$$

this is the identical EE shown in the SPP:[8]

$$\overbrace{(1 + r_{t+1} - \delta)}^{f'(k_{t+1})} \beta U'(c_{t+1}) = \overbrace{U'(c_t)}^{U'(f(k_t) - k_{t+1})}$$

---

[8] $r_t = F_k(k_t, 1) = f'(k_t) - (1-\delta) \quad \Rightarrow \quad f'(k_t) = r_t + (1-\delta)$

also the TVC is:

$$\lim_{t\to\infty} p_t k_{t+1} = 0$$

using the EE from the SPP, this condition is shown equivalent to the one in the SPP:

$$\lim_{t\to\infty} p_t k_{t+1} = \frac{1}{\mu}\lim_{t\to\infty} \beta^t U'(c_t)k_{t+1} = \frac{1}{\mu}\lim_{t\to\infty} \beta^t U'(f(k_t) - k_{t+1})f'(k_t)k_{t+1} = 0$$

then we have the <u>same EE and TVC</u> which means that $\{k_{t+1}\}$ is a P.O. if it is a Competitive Equilibrium allocation.

After the solution is found the rest of the equilibrium can be tracked down easily:

$c_t = f(k_t) - k_{t+1}$
$y_t = F(k_t, 1)$
$i_t = y_t - c_t$
$n_t = 1$

the same with factor prices:

$r_t = F_k(k_t, 1)$ and $w_t = F_n(k_t, 1)$

finally prices are derived from the EE given $p_0 = 1$:

$$\frac{p_{t+1}}{p_t}\frac{p_t}{p_{t-1}}\frac{p_{t-1}}{p_{t-2}}\cdots\frac{p_1}{p_0} = \beta^{t+1}\frac{U'(c_{t+1})}{U'(c_t)}\frac{U'(c_t)}{U'(c_{t-1})}\cdots\frac{U'(c_1)}{U'(c_0)}$$

$$p_{t+1} = \beta^{t+1}\frac{U'(c_{t+1})}{U'(c_0)} = \prod_{\tau=0}^{t}\frac{1}{1+r_{\tau+1}-\delta}$$

Sequential Markets Equilibrium

Prices $\{w_t, r_t\}_{t=0}^{\infty}$, HH allocations $\{c_t, k^s_{t+1}\}$ and firm allocations $\{n^d_t, k^d_t\}$ s.t.

1. Given k_0 and prices $\{w_t, r_t\}_{t=0}^{\infty}$, the allocation $\{c_t, k^s_{t+1}\}$ solves the sequential HH-UMP:

$$\max_{\{c_t, k_{t+1}\}} \sum_{t=0}^{\infty} \beta^t U(c_t)$$

$$\text{s.t.} \qquad c_t + k_{t+1} - (1-\delta)k_t \leq r_t k_t + w_t$$

$$c_t, k_{t+1} \geq 0$$

$$k_0 \quad \text{given}$$

2. Given prices $\{w_t, r_t\}_{t=0}^{\infty}$, for each $t \geq 0$, the firm allocation $\{n^d_t, k^d_t\}$ solves the PMP:

$$\max_{k_t, n_t \geq 0} F(k_t, n_t) - w_t n_t - r_t k_t$$

3. MC: for all $t \geq 0$

$$n_t^d = 1$$
$$k_t^d = k_t^s$$
$$F(k_t^d, n_t^d) = c_t + k_{t+1}^s - (1-\delta)k_t^s$$

Recursive competitive equilibrium

In general the SPP needs to be solved recursively. In models where the equilibrium is not P.O. (some of the assumptions of the welfare theorems is not satisfied) is necessary to use a recursive framework via the Bellman equation.

14.1 General NGM setup with leisure-labor tradeoff and population growth

- In general as stated before the model can include non stationary variables. All we need is to re state the problem in terms of **efficiency variables** (stationary or growth deflated).

Competitive equilibrium

sequences $\{c_t, h_t^s, k_{t+1}^s, N_{t+1}, K_t^d, H_t^d\}_{t=0}^{\infty}$ and prices $\{p_t, r_t, w_t\}_{t=0}^{\infty}$ such that:

1. Given the prices $\{p_t, r_t, w_t\}_{t=0}^{\infty}$ the sequences prices $\{c_t, h_t^s, k_{t+1}^s\}_{t=0}^{\infty}$ solve the HH-UMP,

$$\max_{\{c_t, k_{t+1}, h_t\}_{t=0}^{\infty}} \sum_{t=0}^{\infty} \beta^t N_t u(c_t, \mathbf{1} - \mathbf{h_t})$$
$$\text{s.t.} \quad \sum_{t=0}^{\infty} p_t \mathbf{N_t}(c_t + (\mathbf{1} + \mathbf{n})k_{t+1}) = \sum_{t=0}^{\infty} p_t \mathbf{N_t}[w_t \mathbf{h_t} + (r_t + 1 - \delta)k_t]$$
$$c_t, k_t > 0, 0 < h_t < 1 \quad \forall t$$

2. Given prices $\{p_t, r_t, w_t\}_{t=0}^{\infty}$, the sequences $\{K_t^d, H_t^d\}_{t=0}^{\infty}$ solve the firm's PMP for each t:

$$\max_{K_t, \mathbf{H_t}} F(K_t, \mathbf{H_t}, t) - r_t K_t - w_t \mathbf{H_t}$$

3. MC:

$$h_t^s N_t = K_t^d$$
$$k_t^s N_t = K_t^d$$
$$c_t N_t + K_{t+1}^s - (1-\delta)K_t^s = F(K_t^d, H_t^d, t)$$

4. Population evolves according to $N_{t+1} = N_t(1+n)$

Solving for the equilibrium conditions: Set the Lagrangian,

$$L = \sum_{t=0}^{\infty} \beta^t N_t u(c_t, 1 - h_t) - \lambda \sum_{t=0}^{\infty} p_t N_t [c_t + k_{t+1} - w_t h_t - (r_t + 1 - \delta)k_t]$$

FOC

$[c_t]$: $\beta^t u_1(c_t, 1-h_t)N_t = \lambda p_t N_t$
$[h_t]$: $-\beta^t u_2(c_t, 1-h_t)N_t = -\lambda p_t N_t w_t$
$[k_{t+1}]$: $\lambda p_t N_t(1+n) = \lambda p_{t+1} N_{t+1}(r_{t+1} + 1 - \delta)$

from these equations we get the intratemporal (H1) and intertemporal (H2) tradeoff equations:

$$u_2(c_t, 1-h_t) = w_t u_1(c_t, 1-h_t) \tag{H1}$$

$$u_1(c_t, 1-h_t) = \beta u_1(c_{t+1}, 1-h_{t+1})(r_{t+1} + 1 - \delta) \tag{H2}$$

Also from the firm's PMP FOC we have:

$$r_t = F_1(K_t, H_t, t) = F_1(k_t, h_t, t) \tag{F1}$$

$$w_t = F_2(K_t, H_t, t) = F_2(k_t, h_t, t) \tag{F2}$$

On the other hand the aggregate goods market MC condition is:

$$C_t + K_{t+1} = F(K_t, H_t, t) + (1-\delta)K_t$$

in per-capita terms (divide by (N_t)):

$$c_t + k_{t+1}(1+n) = F(k_t, h_t, t) + (1-\delta)k_t \tag{M}$$

Finally the remaining condition is the TVC:

$$\begin{aligned} \lim_{t\to\infty} p_t k_{t+1} &= 0 \\ \lim_{t\to\infty} p_{t-1} k_t &= 0 \\ \lim_{t\to\infty} \frac{\beta^{t-1} u_1(c_{t-1}, 1-h_{t-1})}{\lambda} k_t &= 0 \\ \lim_{t\to\infty} \frac{\beta^t u_1(c_t, 1-h_t)(r_t + 1 - \delta)}{\lambda} k_t &= 0 \end{aligned} \tag{TVC}$$

the equilibrium would be a path that satisfies $[H1], [H2], [F1], [F2], [M], [TVC]$

Solving for the equilibrium

Suppose the following functional forms

$$\begin{aligned} u(c_1, 1-h_t) &= \ln c_t + \eta \ln(1-h_t) \\ F(k_t, h_t, t) &= A k_t^{\alpha}((1+g)^t h_t)^{1-\alpha} \end{aligned}$$

the equilibrium conditions becomes:

$$\frac{\eta}{1-h_t} = \frac{w_t}{c_t} \qquad \text{[H1]}$$
$$\frac{1}{c_t} = \frac{\beta}{c_{t+1}}(r_{t+1}+1-\delta) \qquad \text{[H2]}$$
$$r_t = A\alpha(1+g)^{t(1-\alpha)}\left(\frac{h_t}{k_t}\right)^{1-\alpha} \qquad \text{[F1]}$$
$$w_t = A(1-\alpha)(1+g)^{t(1-\alpha)}\left(\frac{k_t}{h_t}\right)^{\alpha} \qquad \text{[F2]}$$
$$c_t + k_{t+1}(1+n) = Ak_t^{\alpha}((1+g)^t h_t)^{1-\alpha} + (1-\delta)k_t \qquad \text{[M]}$$

Proposition: most of the variables grow at the same rate and the rest do not grow. That is: $1+\gamma_c = 1+\gamma_k = 1+\gamma_w = 1+\gamma_y = 1+g$ and $\gamma_r = \gamma_h = 0$

Proof: Assume the proposition is correct (balanced growth - BG) then the equations can be writen as:

$$\eta c_t = w_t - h_t w_t \qquad [H1^{BG}]$$
$$1+\gamma_c = \beta(r_{t+1}+1-\delta) \qquad [H2^{BG}]$$
$$1+\gamma_r = \left(\frac{(1+g)(1+\gamma_h)}{1+\gamma_k}\right)^{1-\alpha} \qquad [F1^{BG}]$$
$$1+\gamma_w = (1+g)^{(1-\alpha)}\left(\frac{1+\gamma_k}{1+\gamma_h}\right)^{\alpha} \qquad [F2^{BG}]$$
$$\frac{c_t}{k_t} + (1+\gamma_k)(1+n) = \frac{r_t}{\alpha} + (1-\delta) \qquad [M^{BG}]$$

where the equations $[F1^{BG}]$, $[F2^{BG}]$ where put in terms of growth rates by dividing the equations in $t+1$ over the corresponding in the t, the equation $[H2^{BG}]$ was rearranged and the equation $[M^{BG}]$ was divided by k_t:

from $[H2^{BG}]$: $\gamma_r = 0$

then from $[F1^{BG}]$: $(1+g)(1+\gamma_h) = 1+\gamma_k$

subs. in $[F2^{BG}]$: $1+\gamma_w = 1+g$

from $[M^{BG}]$: the RHS is still ($\gamma_r = 0$) and then the LHS too which implies $\frac{c_t}{k_t}$ is constant through t then $\gamma_c = \gamma_k$

also $[H1^{BG}]$ implies that $\gamma_h = 0$ since it should hold that $1+\gamma_c = 1+\gamma_w = (1+\gamma_w)(1+\gamma_h)$

finally from $y_t = Ak_t^{\alpha}((1+g)^t h_t)^{1-\alpha}$ in $t, t+1$: $1+\gamma_y = (1+\gamma_k)^{\alpha}((1+g)(1+\gamma_h))^{1-\alpha} = 1+g$ ■

This property (all variables growing at the same positive rate or being still) can be used to redefine the model and using the standard procedure for the model with $g = 0$.

Re-define the variables as efficiency variables dividing by $(1+g)^t$:

$$\begin{aligned}
\hat{c}_t &= \frac{c_t}{(1+g)^t} \\
\hat{k}_t &= \frac{k_t}{(1+g)^t} \\
\hat{w}_t &= \frac{w_t}{(1+g)^t} \\
\hat{h}_t &= h_t \\
\hat{r}_t &= r_t
\end{aligned}$$

the equilibrium conditions in terms of the efficiency variables are:

$$\frac{\eta}{1-\hat{h}_t} = \frac{\hat{w}_t}{\hat{c}_t} \tag{H1}$$

$$\frac{\hat{c}_{t+1}(1+g)}{\beta\hat{c}_t} = \hat{r}_{t+1} + 1 - \delta \tag{H2}$$

$$\hat{r}_t = A\alpha\left(\frac{\hat{h}_t}{\hat{k}_t}\right)^{1-\alpha} \tag{F1}$$

$$\hat{w}_t = A(1-\alpha)\left(\frac{\hat{k}_t}{\hat{h}_t}\right)^{\alpha} \tag{F2}$$

$$\hat{c}_t + \hat{k}_{t+1}(1+g)(1+n) = A\hat{k}_t^{\alpha}\hat{h}_t^{1-\alpha} + (1-\delta)\hat{k}_t \tag{M}$$

here the efficiency variables will converge to an steady state and therefore the original per-capita variables converge to a BGP.

Proposition: $\hat{k}_t$ cannot grow unboundedly.

Proof: (by contradiction) Suppose not, then $\hat{k}_t \to \infty$.
since $\hat{h}_t < 1$ then from (F1) $\hat{r}_t \to 0$
in (H2) it implies that $\frac{\hat{c}_{t+1}}{\hat{c}_t} = \frac{\beta(1-\delta)}{1+g} < 1$ then the consumption will be shrinking
finally divide (M) by $\hat{k}_t(1+g)(1+n)$: $\frac{\hat{c}_t}{\hat{k}_t}\frac{1}{(1+g)(1+n)} + \frac{\hat{k}_{t+1}}{\hat{k}_t} = A\left(\frac{\hat{h}_t}{\hat{k}_t}\right)^{1-\alpha}\frac{1}{(1+g)(1+n)} + \frac{1-\delta}{(1+g)(1+n)}$
in this expression the RHS $\to \frac{1-\delta}{(1+g)(1+n)}$ whereas the LHS $\to \infty$ $\quad \to\leftarrow$ ■

Steady State

to find the SS we solve the system of equations:

$$\frac{\eta}{1-\hat{h}_{ss}} = \frac{\hat{w}_{ss}}{\hat{c}_{ss}} \tag{H1}$$

$$(1+g) = \hat{r}_{ss} + 1 - \delta \tag{H2}$$

$$\hat{r}_{ss} = A\alpha\left(\frac{\hat{h}_{ss}}{\hat{k}_{ss}}\right)^{1-\alpha} \tag{F1}$$

$$\hat{w}_t = A(1-\alpha)\left(\frac{\hat{k}_{ss}}{\hat{h}_{ss}}\right)^{\alpha} \tag{F2}$$

$$\hat{c}_{ss} + \hat{k}_{ss}((1+g)(1+n) - 1 + \delta) = A\hat{k}_{ss}^{\alpha}\hat{h}_{ss}^{1-\alpha} \tag{M}$$

in this case we can find the solution analitically. Subs. (F1) into (H2) to get $\hat{h}_{ss}$ in terms of $\hat{k}_s s$:

$$1+g=\beta\left(A\alpha\left(\frac{\hat{h}_{ss}}{\hat{k}_{ss}}\right)^{1-\alpha}+1-\delta\right)$$

$$\hat{h}_{ss}=\underbrace{\left[\left(\tfrac{1+g}{\beta}-1+\delta\right)\tfrac{1}{A\alpha}\right]^{\frac{1}{1-\alpha}}}_{\varepsilon_1}\hat{k}_{ss}=\varepsilon_1\hat{k}_{ss}$$

replace in (H1) and also use (F2) (wage):

$$\begin{aligned}\hat{c}_{ss}&=\frac{(1-\hat{h}_{ss})}{\eta}\hat{w}_{ss}\\&=\frac{(1-\hat{h}_{ss})}{\eta}\hat{(}1-\alpha)A\left(\frac{\hat{k}_{ss}}{\cancelto{\varepsilon_1\hat{k}_{ss}}{\hat{h}_{ss}}}\right)\\&=(1-\varepsilon_1\hat{k}_{ss})\underbrace{\frac{(1-\alpha)A}{\eta\varepsilon_1^{\alpha}}}_{\varepsilon_2}=(1-\varepsilon_1\hat{k}_{ss})\varepsilon_2\end{aligned}$$

subs. in (M):

$$\begin{aligned}(1-\varepsilon_1\hat{k}_{ss})\varepsilon_2+\hat{k}_{ss}((1+g)(1+n)-1+\delta)&=A\varepsilon_1^{1-\alpha}\hat{k}_{ss}\\\hat{k}_{ss}&=\frac{\varepsilon_2}{1-\delta+A\varepsilon_1^{1-\alpha}+\varepsilon_1\varepsilon_2-(1+g)(1+n)}\end{aligned}$$

the parameters for this economy are $\{\beta,\eta,\alpha,A,g,\delta,n\}$ with numerical values for these the economy can be find for the assumed functional form.

Example: define the economy by the functional forms $u(c_t,1-h_t)=\ln c_t+\eta\ln(1-h_t)$ and $F(k_t,h_t,t)=Ak_t^{\alpha}((1+g)^th_t)^{1-\alpha}$ and $A=1$, $\delta=0.015$, $\eta=1$, $g=0.005$, $\alpha=0.3$, $\beta=0.99$, $n=0.001$.

then:

$$\begin{aligned}\varepsilon_1&=\left[\left(\tfrac{1+g}{\beta}-1+\delta\right)\tfrac{1}{A\alpha}\right]^{\frac{1}{1-\alpha}}=\left[\left(\tfrac{1+.005}{.99}-1+.005\right)\tfrac{1}{.3}\right]^{\frac{1}{1-.3}}=.0375\\\varepsilon_2&=\frac{(1-\alpha)A}{\eta\varepsilon_1^{\alpha}}=\frac{1-.3}{.0375^{.3}}=1.874\\\hat{k}_{ss}&=\frac{1.874}{1-.015+.0375^{1-.3}+(.0375)(1.874)-(1.005)(1.001)}=12.505\\\hat{c}_{ss}&=(1-\varepsilon_1\hat{k}_{ss})\varepsilon_2=(1-.375(12.505))1.874=.994\\\hat{h}_{ss}&=\varepsilon_1\hat{k}_{ss}=.375(12.505)=.469\\\hat{r}_{ss}&=A\alpha\left(\frac{\hat{h}_{ss}}{\hat{k}_{ss}}\right)^{1-\alpha}=.3\left(\frac{.469}{12.505}\right)^{1-.3}=.03\\\hat{w}_{ss}&=A(1-\alpha)\left(\frac{\hat{k}_{ss}}{\hat{h}_{ss}}\right)^{\alpha}=(1-.3)\left(\frac{12.505}{.469}\right)^{.}3=1.874\end{aligned}$$

additionally the balanced growth path per-capita variables are:

$$\begin{aligned}
k_t^{BGP} &= \hat{k}_{ss}(1+g)^t = 12.505(1.005)^t \\
c_t^{BGP} &= \hat{c}_{ss}(1+g)^t = .994(1.005)^t \\
w_t^{BGP} &= \hat{w}_{ss}(1+g)^t = 1.874(1.005)^t \\
h_t^{BGP} &= \hat{h}_{ss} = 0.469 \\
r_t^{BGP} &= \hat{r}_{ss} = 0.03
\end{aligned}$$

14.2 Welfare analysis

Building on the former results we can compare different growth paths and apply welfare comparisons after different shocks with respect to a benchmark in terms of a scale factor of lifetime consumption. The former will be our measure of welfare to determine in which economies there are better life standards.

Solving for lifetime utility implied by the BGP

Assuming we are on the BGP since the initial period:

$$\begin{aligned}
U_{BGP} &= \sum_{t=0}^{\infty} \beta^t N_t \left[\ln c_t^{BGP} + \eta \ln(1 - h_t^{BGP})\right] \\
&= \sum_{t=0}^{\infty} \beta^t N_0 (1+n)^t \left[\ln(\hat{c}_{ss}(1+g)^t) + \eta \ln(1 - \hat{h}_{ss})\right] \\
&= N_0 \sum_{t=0}^{\infty} (\beta(1+n))^t \left[\ln \hat{c}_{ss} + t \ln(1+g) + \eta \ln(1 - \hat{h}_{ss})\right] \\
&= N_0 \left(\ln \hat{c}_{ss} + \eta \ln(1 - \hat{h}_{ss})\right) \sum_{t=0}^{\infty} (\beta(1+n))^t + N_0 \ln(1+g) \sum_{t=0}^{\infty} t(\beta(1+n))^t \\
&= N_0 \left(\ln \hat{c}_{ss} + \eta \ln(1 - \hat{h}_{ss})\right) \frac{1}{1-\beta(1+n)} + N_0 \ln(1+g) \frac{\beta(1+n)}{[1-\beta(1+n)]^2}
\end{aligned}$$

Solving for lifetime utility implied by any equilibrium path

For any other path we just need to partitionate the periods in a part in which the variables haven't reached the BGP (denote this part A) and the remaining periods when the economy has already converged to the BGP. Assume t^* is a period in which the economy already has converged to its BGP:

$$\begin{aligned}
U &= \sum_{t=0}^{\infty} \beta^t N_t [\ln c_t + \eta \ln(1 - h_t)] \\
&= \sum_{t=0}^{t^*-1} [\ln c_t + \eta \ln(1 - h_t)] + \sum_{t=t^*}^{\infty} \beta^t N_t [\ln c_t + \eta \ln(1 - h_t)] \\
&= A + B
\end{aligned}$$

$$
\begin{aligned}
B &= \sum_{t=t^*}^{\infty} \beta^t N_t [\ln c_t + \eta \ln(1-h_t)] \\
&= \sum_{t=t^*}^{\infty} \beta^t N_0 (1_n)^t [\ln(\hat{c}_{ss}(1+g)^t) + \eta \ln(1-\hat{h}_{ss})] \\
&= N_0 \sum_{t=t^*}^{\infty} (\beta(1+n))^t \left[\ln \hat{c}_{ss} + t\ln(1+g) + \eta \ln(1-\hat{h}_{ss})\right] \\
&= N_0 \left(\ln \hat{c}_{ss} + \eta \ln(1-\hat{h}_{ss})\right) \sum_{t^*}^{\infty} (\beta(1+n))^t + N_0 \ln(1+g) \sum_{t^*}^{\infty} t(\beta(1+n))^t \\
&= N_0 \left(\ln \hat{c}_{ss} + \eta \ln(1-\hat{h}_{ss})\right) \frac{(\beta(1+n)^{t^*})}{1-\beta(1+n)} + N_0 \ln(1+g) \left[\frac{(\beta(1+n))^{t^*}}{(1-\beta(1+n))^2} + \frac{(t^*-1)(\beta(1+n))^{t^*}}{1-\beta(1+n)}\right]
\end{aligned}
$$

Example: Increase of 30% in productivity

Assume the Japanese economy is described by the following parametrization: $A = 1, \delta = 0.05, \eta = 1.44, g = 0.02, \theta = 0.3, \beta = 0.9, n - 0, N_0 = 1, k_0 = \hat{k}_{ss}$.

Suppose the economy is subject to a 30% increase in its productivity, then $A_{new} = 1.3$

1. Compute the initial SS:

Original Steady State (A=1)

$\hat{k}_{ss}$	$\hat{c}_{ss}$	$\hat{h}_{ss}$
0.716	0.388	0.354

this implies that,

$$
\begin{aligned}
c_t^{BGP} &= \hat{c}_{ss}(1+g)^t = 0.388(1.02^t) \\
h_t^{BGP} &= \hat{h}_{ss} = 0.3544 \\
N_t = N_0(1+n)^t &= 1(1+0)^t = 1
\end{aligned}
$$

also the lifetime utility in the BGP is given by:

$$
\begin{aligned}
U^{BGP} &= N_0 \left(\ln \hat{c}_{ss} + \eta \ln(1-\hat{h}_{ss})\right) \frac{1}{1-\beta(1+n)} + N_0 \ln(1+g) \frac{\beta(1+n)}{[1-\beta(1+n)]^2} \\
U^{BGP} &= (\ln 0.388 + 1.44 \ln(1-0.3544)) \frac{1}{1-.9} + \ln(1.02) \frac{.9}{[1-.9]^2} = -13.997
\end{aligned}
$$

2. Compute the lifetime utility of the economy when the shock is observed (A=1.30)

New Steady State (A=1)

$\hat{k}_{ss}$	$\hat{c}_{ss}$	$\hat{h}_{ss}$
1.042	0.564	0.354

Remember that the initial value for capital is still $k_0 = 0.7162$ (the old initial BGP value).

we find manually the initial part: $A = U_{t=0}^{t=30} = -10.8232$

and for B we use our formula:

$$B = U_{t=31}^{t=\infty} = (\ln 0.564 + 1.44 \ln(1-0.354))\frac{.9^{31}}{1-.9} + (\ln 1.02)\left(\frac{.9^{31}}{(1-.9)^2} + \frac{30(.9^31)}{1-.9}\right) = -0.1568$$

then the lifetime utility of the economy with the shock is:

$$U^{A=1.30} = A + B + -10.8232 - 0.1568 = -10.98$$

which means that the economy had an improvement given his utility in the former state was $U^{BGP} = -13.997$

3. **Welfare analysis**: Calculate the scale factor ϕ of lifetime consumption in the original economy that would equate the new utility after the shock.

what we look is for ϕ such that $U^{A=1}\left(\phi c_t^{BGP}, h_t^{BGP}\right) = U^{A=1.30}$,

$$\sum_{t=0}^{\infty} \beta^t \left[\ln(\phi c_t^{BGP}) + \eta \ln(1 - h_t^{BGP})\right] = U^{A=1.30}$$

$$\sum_{t=0}^{\infty} \beta^t \left[\ln(\phi \hat{c}_{ss}(1+g)^t) + \eta \ln(1 - \hat{h}_{ss})\right] = U^{A=1.30}$$

$$\sum_{t=0}^{\infty} 0.9^t \left[\ln(\phi(0.388)1.02^t) + 1.44 \ln(1 - 0.3544)\right] = -10.98$$

$$\sum_{t=0}^{\infty} 0.9^t \left[\ln\phi + \ln(0.388) + t\ln 1.02 + 1.44 \ln(1 - 0.3544)\right] = -10.98$$

$$\sum_{t=0}^{\infty} 0.9^t \left[\ln\phi + t\ln 1.02 - 1.578\right] = -10.98$$

$$\sum_{t=0}^{\infty} 0.9^t \left[\ln\phi + t\ln 1.02 - 1.578\right] = -10.98$$

$$(\ln\phi - 1.578)\frac{1}{1-0.9} + (\ln 1.02)\frac{0.9}{(1-0.9)^2} = -10.98$$

$$\phi = 1.352$$

therefore, an increase in the productivity of 30% implies a welfare improvement equivalent to increasing the agents lifetime consumption 35%.

14.3 NGM model with taxes

Government and taxes are introduced to the NGM model. The taxes are the following: tax on labor income τ_{ht}, tax on capital income τ_{kt}, tax on consumption spending τ_{ct}, tax on investment expenditures τ_{xt} and lump-sum taxes T. On the other hand the goverment consumes g_t (gov. spending per-capita) and the remainder is distributed to the HH via transfers (lump-sum taxes). Hence the government BC is:

$$\sum_{t=0}^{\infty} p_t(g_t + T_t) = \sum_{t=0}^{\infty} p_t(\tau_{ct}c_t + \tau_{xt}x_t + \tau_{kt}r_tk_t + \tau_{ht}w_th_t) \qquad [BC^g]$$

Notice the BC is writen as `Gov Transfers to HH = Gov Revenues`

HH UMP

taking fiscal policy as given, the HH solves the following UMP:

$$\max_{\{c_t,x_t,k_{t+1},h_t\}} \sum_{t=0}^{\infty} \beta^t u(c_t, 1-h_t)$$

$$\text{s.t.} \quad \sum_{t=0}^{\infty} p_t[(1+\tau_{ct})c_t + (1+\tau_{xt})x_t] = \sum_{t=0}^{\infty} p_t[(1-\tau_{ht})w_t h_t + (1-\tau_{kt})r_t k_t + T_t] \qquad [BC^{HH}]$$

$$k_{t+1} = (1-\delta)k_t + x_t \qquad [\text{CA}]$$

$$c_t, k_t > 0, 0 < h_t < 1$$

set the lagrangian:

$$L = \sum_{t=0}^{\infty} \beta^t u(c_t, 1-h_t) - \lambda \sum_{t=0}^{\infty} p_t[(1+\tau_{ct})c_t + (1+\tau_{xt})x_t - (1-\tau_{ht})w_t h_t - (1-\tau_{kt})r_t k_t - T_t] - \boldsymbol{\beta^t \mu}_t(k_{t+1} - (1-\delta)k_t - x_t)$$

FOC:

$$\begin{aligned}
&[c_t]: \quad \beta^t u_1(c_t, 1-h_t) = \lambda(1+\tau_{ct})p_t \\
&[h_t]: \quad \beta^t u_2(c_t, 1-h_t) = \lambda(1-\tau_{ht})p_t w_t \\
&[x_t]: \quad -\lambda p_t(1+\tau_{xt}) + \beta^t \mu_t = 0 \\
&[k_{t+1}]: \quad \lambda p_{t+1}(1-\tau_{kt+1})r_{t+1} - \beta^t \mu_t + \beta^{t+1}\mu_{t+1}(1-\delta) = 0
\end{aligned}$$

replace $\beta^t \mu_t$ and $\beta^{t+1}\mu_{t+1}$ from $[x_t]$ in $[k_{t+1}]$:

$$\frac{p_t}{p_{t+1}} = \frac{1}{1+\tau_{xt}}[(1-\tau_{kt+1})r_{t+1} + (1-\tau_{xt+1})(1-\delta)]$$

this is known as the no arbitrage condition: relative price of capital = return on saving

Combine the first two FOC:

$$\frac{u_2(c_t, 1-h_t)}{u_1(c_t, 1-h_t)} = \frac{(1-\tau_{ht})w_t}{(1-\tau_{ct})} \qquad [\text{H1}]$$

Also use the first FOC in t and t+1 as well as the no arbitrage condition to get the intertemporal trade-off (Euler Eq.):

$$\begin{aligned}
\frac{u_1(c_t, 1-h_t)}{\beta u_1(c_{t+1}, 1-h_{t+1})} &= \frac{(1+\tau_{ct})}{(1+\tau_{ct+1})} \frac{p_t}{p_{t+1}} \\
\frac{u_1(c_t, 1-h_t)}{\beta u_1(c_{t+1}, 1-h_{t+1})} &= \frac{(1+\tau_{ct})}{(1+\tau_{ct+1})} \frac{1}{1+\tau_{xt}}[(1-\tau_{kt+1})r_{t+1} + (1-\tau_{xt+1})(1-\delta)] \\
u_1(c_t, 1-h_t)\frac{(1+\tau_{xt})}{(1+\tau_{ct})} &= \frac{\beta u_1(c_{t+1}, 1-h_{t+1})}{(1+\tau_{ct+1})}[(1-\tau_{kt+1})r_{t+1} + (1+\tau_{xt+1})(1-\delta)] \qquad [\text{H2}]
\end{aligned}$$

Also the prices are pinned down from the consumption FOC in periods t and 0:

$$\frac{\beta^t u_1(c_t, 1-h_t)}{u_1(c_0, 1-h_0)} = \frac{(1+\tau_{ct})}{(1+\tau_{c0})}\frac{p_t}{p_0}$$

normalize by $p_0 = 1$ and then,

$$p_t = \frac{\beta^t u_1(c_t, 1-h_t)}{u_1(c_0, 1-h_0)}\frac{(1+\tau_{c0})}{(1+\tau_{ct})} \qquad \text{[P]}$$

Firm's PMP

the firm maximizes profits as before, yielding the same equilibrium conditions:

$$\max_{K_t, H_t} F(K_t, H_t, t) - r_t K_t - w_t H_t$$

FOC:

$$r_t = F_1(K_t, H_t, t) = F_1(k_t, h_t, t) \qquad \text{[F1]}$$
$$w_t = F_2(K_t, H_t, t) = F_2(k_t, h_t, t) \qquad \text{[F2]}$$

The goods market MC condition is obtained by subs. the taxes from the government into the BC for the HH:

$$c_t + x_t + \mathbf{g_t} = F(k_t, h_t) \qquad \text{[MC]}$$

The former optimization problems and fiscal policy form the Tax Distorted Competitive Equilibrium (TDCE), an important result is that the TDCE resulting allocation is inefficient, this is because the taxes distort the relative prices and agents optimal decisions.

Tax Distorted Competitive Equilibrium TDCE

for a given fiscal policy $\{\tau_{ct}, \tau_{ht}, \tau_{xt}, \tau_{kt}, T_t, G_t\}$ the TDCE is given by the sequences $\{c_t, h_t^S, k_{t+1}^s, K_t^d, H_t^d\}_{t=0}^{\infty}$ and $\{p_t, r_t, w_t\}_{t=0}^{\infty}$ such that:

1. Given $\{p_t, r_t, w_t\}$, the sequences $\{c_t, h_t^s, k_{t+1}^s\}$ solves the HH UMP.
2. Given $\{p_t, r_t, w_t\}$, the sequences (K_t^D, H_t^D) solve the firm's PMP for each t.
3. All markets clear (MC) for each t:

$$h_t^s N_t = H_t^D$$
$$k_t^s N_t = K_t^D$$
$$c_t^s N_t + K_{t+1}^s - (1-\delta)K_t^s + G_t = F(K_t^d, H_t^d, t)$$

Lemma: A TDCE doesn't exist for all fiscal policies

Proof: E.g. $g_t > 0 \quad \forall t$ but all tax rates (including Lump-sum) are zero. In such case the government's BC cannot be balanced. ■

Lemma: $[BC^{HH}]$, $[MC] \quad \Rightarrow \quad [BC^g]$, also $[BC^g]$, $[MC] \quad \Rightarrow \quad [BC^{HH}]$

Proof: Expand taxes in $[BC^{HH}]$,

$$\begin{aligned}
\sum_{t=0}^{\infty} p_t[c_t+\tau_{ct}c_t+x_t+\tau_{xt}x_t] &= \sum_{t=0}^{\infty} p_t[w_th_t-\tau_{ht}w_th_t+r_tk_t-\tau_{kt}r_tk_t+T_t] \\
\sum_{t=0}^{\infty} p_t[c_t+x_t+\tau_{ct}c_t+\tau_{xt}x_t] &= \sum_{t=0}^{\infty} p_t[\underbrace{F(k_t,w_t)}_{w_th_t+r_tk_t}-\tau_{ht}w_th_t-\tau_{kt}r_tk_t+T_t] \\
\sum_{t=0}^{\infty} p_t[\underbrace{F(k_t,h_t)-g_t}_{c_t+x_t \quad \text{(in [MC])}}+\tau_{ct}c_t+\tau_{xt}x_t] &= \sum_{t=0}^{\infty} p_t[F(k_t,w_t)-\tau_{ht}w_th_t-\tau_{kt}r_tk_t+T_t] \\
\sum_{t=0}^{\infty} p_t[\cancel{F(k_t,w_t)}+\tau_{ct}c_t+\tau_{xt}x_t] &= \sum_{t=0}^{\infty} p_t[\cancel{F(k_t,w_t)}+g_t-\tau_{ht}w_th_t-\tau_{kt}r_tk_t+T_t] \\
\sum_{t=0}^{\infty} p_t[\tau_{ct}c_t+\tau_{xt}x_t+\tau_{ht}w_th_t+\tau_{kt}r_tk_t] &= \sum_{t=0}^{\infty} p_t[g_t+T_t] \qquad [BC^g]
\end{aligned}$$

Summary: the equilibrium paths in a TDCE must satisfy the following conditions:

$$\begin{aligned}
\tfrac{u_2(c_t,1-h_t)}{u_1(c_t,1-h_t)} &= \tfrac{(1-\tau_{ht})w_t}{(1+\tau_{ct})} && \text{[H1]} \\
u_1(c_t,1-h_t)\tfrac{(1+\tau_{xt})}{(1+\tau_{ct})} &= \tfrac{\beta}{(1+\tau_{ct+1})}u_1(c_{t+1},1-h_{t+1})[(1-\tau_{kt+1})r_{t+1}+(1+\tau_{xt+1})(1-\delta)] && \text{[H2]} \\
r_t &= F_1(k_t,h_t) && \text{[F1]} \\
w_t &= F_2(k_t,h_t) && \text{[F2]} \\
c_t+x_t+g_t &= F(k_t,h_t) && \text{[MC]} \\
k_{t+1} &= (1-\delta)k_t+x_t && \text{[CA]} \\
p_t &= \tfrac{\beta^t u_1(c_t,1-h_t)}{u_1(c_0,1-h_0)}\frac{(1+\tau_{c0})}{(1+\tau_{ct})} && \text{[P]} \\
\sum_{t=0}^{\infty} p_t[(1+\tau_{ct})c_t+(1+\tau_{xt})x_t] &= \sum_{t=0}^{\infty} p_t[(1-\tau_{ht})w_th_t+(1-\tau_{kt})r_tk_t+T_t] && [BC^{HH}]
\end{aligned}$$

as seen bellow the last equation can be simplified to:

$$\frac{(1+\tau_{c0})}{u_1(c_0,1-h_0)}\sum_{t=0}^{\infty}\beta^t[u_1(c_t,1-h_t)c_t-u_2(c_t,1-h_t)h_t] = (1-\tau_{k0})r_0k_0+(1-\delta)(1+\tau_{x0})k_0+\sum_{t=0}^{\infty}p_tT_t \quad [BC^{HH}_{FOCs}]$$

we can simplify the $[BC^{HH}]$ condition by replacing other FOCs in it. This will get us closer to state the implementability constraint: a BC compatible with the TDCE.

Substitute x_t (from $[CA]$) in $[BC^{HH}]$ and rearrange:

$$\sum p_t[(1+\tau_{ct})c_t-(1-\tau_{ht})w_th_t)] = \sum p_t[-(1+\tau_{xt})(k_{t+1}-(1-\delta)k_t)+(1-\tau_{kt})r_tk_t+T_t]$$

let us rearrange both sides of this expression:

For the LHS replace p_t from [P] and $(1-\tau_{ht})w_t$ from [H1]

$$
\begin{aligned}
LHS &= \sum p_t[(1+\tau_{ct})c_t - (1-\tau_{ht})w_t h_t] \\
&= \sum \underbrace{\tfrac{\beta^t u_1(c_t, 1-h_t)}{u_1(c_0, 1-h_0)} \frac{(1+\tau_{c0})}{(1+\tau_{ct})}}_{p_t} [(1+\tau_{ct})c_t - (1-\tau_{ht})w_t h_t] \\
&= \sum \frac{\beta^t u_1(c_t, 1-h_t)}{u_1(c_0, 1-h_0)} \frac{(1+\tau_{c0})}{(1+\tau_{ct})} \Big[(1+\tau_{ct})c_t - \underbrace{\frac{u_2(c_t, 1-h_t)(1+\tau_{ct})}{u_1(c_t, 1-h_t)}}_{(1-\tau_{ht})w_t} h_t\Big] \\
&= \sum \frac{\beta^t (1+\tau_{c0})}{u_1(c_0, 1-h_0)} \Big[u_1(c_t, 1-h_t)c_t - u_2(c_t, 1-h_t)h_t\Big]
\end{aligned}
$$

The RHS can be expanded:

$$
\begin{aligned}
RHS = \sum_{t=0}^{\infty} & p_t[-(1-\tau_{xt})(k_{t+1}-(1-\delta)k_t) + (1-\tau_{kt})r_t k_t + T_t] \\
= & \, p_0(1-\tau_{x0})r_0 k_0 + p_0(1-\delta)(1+\tau_{x0})k_0 + [(-p_0(1+\tau_{x0}) + p_1(1+\tau_{k1})r_1 + p_1(1-\tau_{x1})(1-\delta))k_1] \\
& + [(-p_1(1+\tau_{x1}) + p_2(1-\tau_{k2})r_2 + p_2(1+\tau_{x2})(1-\delta))k_2] \\
& + [(-p_2(1+\tau_{x2}) + p_3(1-\tau_{k3})r_3 + p_3(1+\tau_{x3})(1-\delta))k_3] \\
& \vdots \\
& + \sum_{t=0}^{\infty} p_t T_t \\
= & \, p_0(1-\tau_{x0})r_0 k_0 + p_0(1-\delta)(1+\tau_{x0})k_0 + \sum_{t=0}^{\infty} p_t T_t
\end{aligned}
$$

notice that all intermediate terms are zero according to the arbitrage (price-ratio) condition:

$$
\tfrac{p_t}{p_{t+1}} = \tfrac{1}{1+\tau_{xt}}[(1-\tau_{kt+1})r_{t+1} + (1+\tau_{xt+1})(1-\delta)] \quad \Rightarrow \quad p_t(1+\tau_{xt}) = p_{t+1}[(1-\tau_{kt+1})r_{t+1} + (1+\tau_{xt+1})(1-\delta)]
$$

Theorem: Ricardian Equivalence

The timing of transfers (via lump-sum taxes) is irrelevant, i.e., that fiscal policies with the same tax rates and differing in the lump-sum transfers, such that $\sum_{t=0}^{\infty} p_t^* T_t = \sum_{t=0}^{\infty} p_t^* \tilde{T}_t$ are equivalent and yield the same TDCE.

Proof: In the equilibrium conditions $\sum_{t=0}^{\infty} p_t T_t$ only appears in one equation, the rest of conditions are the same in both policies and then T_t doesn't change the allocations, we just need the present value of the transfers to be equal.[9] ■

In general **to prove the Fiscal Policy is implementable**:

1. Look for $\{c_t, h_t, k_t\}$ that satisfy $[H1].[H2],[MC]$
2. Get r_t, w_t from $[F1],[F2]$
3. Get x_t from $[CA]$

---

[9] this holds partly because the consumers are identical and infinite lived, also because credit markets work perfectly in the model (consumers save and borrow as much as they want), with borrowing constraints the results would matter.

4. Get p_t from $[P]$
5. Back out $\sum p_t T_t$ from $[BC^{HH}]$

Of course, the order of the steps matter and reflects that the timing of the transfers is irrelevant (final step).

Proposition: Lump-sum taxes T_t are non distortionary: Assume $\tau_{ct} = \tau_{xt} = \tau_{kt} = \tau_{ht} = 0$ so only T_t is used to finance g_t. Then the TDCE allocation solves:

$$\begin{aligned} &\max \sum_{t=0}^{\infty} \beta^t u(c_t, 1-h_t) \\ \text{s.t.} \qquad & c_t + k_{t+1} - (1-\delta)k_t = \underbrace{\hat{F}(k_t, h_t)}_{F(k_t,h_t)-g_t} \quad \forall t \\ & k_0 \text{ given} \end{aligned}$$

Proof: (Sketch) Notice that given the proportional taxes are not present in general then the FOC of both problems, characterizing the unique solution due to the strict concavity of the utility and production function, would be the same and will support the same allocation. The proportional taxes don't enter in the price ratio equations so are not distortionary.

Proposition: Assume $\tau_{ct} = \tau_{xt} = T_t = 0$ and $\tau_{kt} = \tau_{wt} = \tau_t > 0$ and $g_t = (w_t h_t + r_t k_t)\tau_t$, i.e., the gov. budget is balanced each period. Then the TDCE allocation solves:

$$\begin{aligned} &\max \sum_{t=0}^{\infty} \beta^t u(c_t, 1-h_t) \\ \text{s.t.} \qquad & c_t + k_{t+1} - (1-\delta)k_t = (1-\tau_t)F(k_t, h_t) \quad \forall t \\ & k_0 \text{ given} \end{aligned}$$

Proof: (sketch) Notice $g_t = \tau_t F(k_t, h_t)$ then the [MC] condition would be the same that in the TDCE and the rest of FOC will be analogous too, supporting the same optimal allocation.

Theorem: There are too many taxes

Consider:

fiscal policy 1: $\tau_{ct}, \tau_{xt}, \tau_{kt}, \tau_{ht}, g_t > 0$ with $T_t = 0$
fiscal policy 2: $\tilde{\tau}_{kt}, \tilde{\tau}_{ht}, g_t > 0$ with $\tilde{T}_t = 0$

Any allocation supported through FP1 can be supported through FP2 (notice the same g_t).

Proof: (sketch) with the FP2 set the taxes $\tilde{\tau}_{kt}, \tilde{\tau}_{ht}$ in a way that the same allocation are supported in the FOC. Given the same g_t is present in both FP then the overall solution is the same.

Theorem: Other FP equivalencies

Consider:

fiscal policy 1: $\tau_{kt}, \tau_{ht}, g_t > 0$ with $T_t = 0$
fiscal policy 2: $\tilde{\tau}_{ct}, \tilde{\tau}_{ht}, g_t > 0$ with $\tilde{T}_t = 0$

1. Any given allocation supported through FP1 can be supported with FP2 and viceversa.
2.

- $\tau_{kt} \to 0 \quad \Leftrightarrow \quad \tilde{\tau}_{ct} \to \tilde{\tau}_c$
- $\tau_{kt} \to \tau_k > 0 \quad \Rightarrow \quad \tilde{\tau}_{ct} \to \infty$

Solving for the Steady State

If we assume that tax rates and government consumption converge to some constant rates, then equilibrium paths converge to a SS. There is no need to assume that transfers are constant.

assume the CD production function and the logaritmitc utility with parameter η as before, then in the SS:

$$\frac{\eta c_{ss}}{1-h_{ss}} = \frac{(1-\tau_h)w_{ss}}{(1+\tau_c)} \qquad [H1]^{ss}$$

$$\frac{1}{\beta} = \frac{1}{(1+\tau_x)}[(1-\tau_k)r_{ss} + (1+\tau_x)(1-\delta)] \qquad [H2]^{ss}$$

$$r_{ss} = A\alpha\left(\frac{h_{ss}}{k_{ss}}\right)^{1-\alpha} \qquad [F1]^{ss}$$

$$w_t = A(1-\alpha)\left(\frac{k_{ss}}{h_{ss}}\right)^{\alpha} \qquad [F2]^{ss}$$

$$c_{ss} + x_{ss} + g = Ak_{ss}^{\alpha}h_{ss}^{1-\alpha} \qquad [MC]^{ss}$$

$$\delta k_{ss} = x_{ss} \qquad [CA]^{ss}$$

$$\frac{p_{t+1}}{p_t} = \beta \quad \Rightarrow \quad p_t = \beta^t \qquad [P]^{ss}$$

$$\sum_{t=0}^{\infty}\beta^t[(1+\tau_{ct})c_t + (1+\tau_{xt})x_t] = \sum_{t=0}^{\infty}\beta^t[(1-\tau_{ht})w_t h_t + (1-\tau_{kt})r_t k_t + T_t] \qquad [BC^{HH}]^{ss}$$

in this system of equations is possible to find an analytic solution just as before:

Subs. (F1) into (H2) $\Rightarrow \quad h_{ss} = \varepsilon_1 k_{ss}$
then from (H1) we get: $c_{ss} = (1-\varepsilon_1 k_{ss})\varepsilon_2$
subs. these results in (MC) and subs. the investment using (CA) $\quad \Rightarrow \quad k_{ss} = \frac{\varepsilon_2 + g}{A\varepsilon_1^{1-\alpha} - \delta + \varepsilon_1\varepsilon_2}$
finally $x_{ss} = \delta k_{ss}$

Welfare analysis

If the economy starts at the BGP, i.e., is in the SS from period t=0 and onwards the lifetime utility is:

$$U^{ss} = \sum_{t=0}^{\infty}(\beta^t \ln c_t + \eta\ln(1-h_t)) = \sum_{t=0}^{\infty}\beta^t(\ln c_{ss} + \eta\ln(1-h_{ss})) = \frac{1}{1-\beta}(\ln c_{ss} + \eta\ln(1-h_{ss}))$$

starting at $k_0 \neq k_{ss}$:

$$U = \sum_{t=0}^{\infty}(\beta^t \ln c_t + \eta\ln(1-h_t)) = \sum_{t=0}^{t^*-1}(\beta^t \ln c_t + \eta\ln(1-h_t)) + \sum_{t=t^*}^{\infty}(\beta^t \ln c_t + \eta\ln(1-h_t)) = A + B$$

A is the utility during the transitional period and is computed numerically whereas B can be obtained analytically as:

$$B = \sum_{t=t^*}^{\infty}(\beta^t \ln c_t + \eta\ln(1-h_t)) = \frac{\beta^{t*}}{1-\beta}(\ln c_{ss} + \eta\ln(1-h_{ss}))$$

14.4 Optimal taxation policy: the Ramsey problem

Ramsey problem refers to the optimization problem of the benevolent tax designer: Given $\{g_t\}_{t=0}^{\infty}$ choose the tax policy to maximize consumer utility, that in turn is determined by the TDCE allocations.

First Best: Use only transfers, i.e., lump-sum taxes given they aren't distortionary.

Second Best: When FB is not possible proportional taxes are applied.

the focus is in τ_{kt}, τ_{ht}, according to the previous results, analyzing this is enough to characterize any other fiscal policy reviewed. Additionally $\tau_{\mathbf{k0}} = 1$, this will be optimal since k_0 is already given and increasing the tax won't affect the allocation decisions on capital, or relative prices for that period.

Setup of problems

$[RP1]:$ Given $g \equiv \{g_t\}_{t=0}^{\infty}$ choose $\tau \equiv \{\tau_{kt}, \tau_{ht}\}_{t=0}^{\infty}$ to maximize:

$$\max \sum_{t=0}^{\infty} \beta^t u(c_t(\tau), 1 - h_t(\tau))$$

s.t. $\{c_t(\tau), h_t(\tau)\}$ are the TDCE allocations with fiscal policy (g, τ)

the equilibrium conditions for this model are:

$$\frac{u_2(c_t, 1-h_t)}{u_1(c_t, 1-h_t)} = (1 - \tau_{ht}) w_t \qquad [\text{H1}]$$

$$u_1(c_t, 1-h_t) = \beta u_1(c_{t+1}, 1-h_{t+1})[(1 - \tau_{kt+1}) r_{t+1} + (1-\delta)] \qquad [\text{H2}]$$

$$r_t = F_1(k_t, h_t) \qquad [\text{F1}]$$

$$w_t = F_2(k_t, h_t) \qquad [\text{F1}]$$

$$c_t + x_t + g_t = F(k_t, h_t) \qquad [\text{MC}]$$

$$k_{t+1} = (1-\delta) k_t + x_t \qquad [\text{CA}]$$

$$p_t = \frac{\beta^t u_1(c_t, 1-h_t)}{u_1(c_0, 1-h_0)} \qquad [\text{P}]$$

$$\frac{1}{u_1(c_0, 1-h_0)} \sum_{t=0}^{\infty} \beta^t [u_1(c_t, 1-h_t) c_t - u_2(c_t, 1-h_t) h_t] = (1 - \underbrace{\tau_{k0}}_{1}) r_0 k_0 + (1-\delta) k_0 \qquad [IC]$$

How to get the equilibrium:

- [MC], [CA], [IC]: provide the allocations, this is easier since they have not taxes
- [P] gives the prices
- [F1], [F2] are used to obtain $\{w_t, r_t\}$
- Finally given allocation and factor prices we use [H1] to solve for τ_{ht} and [H2] for τ_{kt+1}

Proposition: An allocation $\{c_t, h_t, x_t, k_{t+1}\}$ is a TDCE for the economy with fiscal policty (g, τ) if and only if [MC], [CA] and [IC] hold for all t.

Proof: $\Rightarrow$ is given by definition. $\Leftarrow$: show there are prices and τ that support $\{c_t^*, h_t^*, x_t^*, k_{t+1}^*\}$ as an equilibrium allocation, i.e., this equations alone pin down the allocations that are also consistent with the whole system.

The latter can be seen in the box above. See how the allocations come from the conditions [MC], [CA], [IC] only. This doesn't mean that the rest are meaningless but that the IC is made in a way that already takes into account the rest of conditions.

Intuition: (i) In principle what we do when solving the RP is: 1. finding optimal the allocation as function of the policy instruments (taxes), this is done by the usual private agents optimization, and, 2. based on that solution, optimizing over the instruments. (ii) the implementability constraint $[IC]$ is a household lifetime budget constraint that already incorporates optimal decisions of private agents that took taxes as given. Therefore, if we solve the planner problem subject to the IC, our solution of the RP2 and RP3 (that are more similar to typical SPP, i.e., max. welfare s.t. feasibility) will be already consistent and compatible with the TDCE.

The former is a loose version of the Ramsey problem, an analogous one, yet more rigorous is:

$[RP2]$:

$$\max_{c_t,x_t,k_{t+1},h_t} \sum_{t=0}^{\infty} \beta^t u(c_t, 1-h_t)$$

$$\text{s.t.} \qquad \text{[MC],[CA],[IC] hold} \quad \forall t$$

this problem, i.e., the RP, is different from the regular SPP because of the [IC] (there we have the standard budget constraint), therefore, the resulting allocation is different.

$$\begin{aligned}
\mathcal{L} &= \sum_{t=0}^{\infty} \beta^t u(c_t, 1-h_t) - \lambda \left(\sum_{t=0}^{\infty} \beta^t [u_1(c_t, 1-h_t)c_t - u_2(c_t, 1-h_t)h_t] - u_1(c_0, 1-h_0)(1-\delta)k_0 \right) + \text{[MC], [CA] terms} \\
&= \underbrace{u(c_0, 1-h_0) - \lambda(u_1(c_0, 1-h_0)c_0 - u_2(c_0, 1-h_0)h_0 - u_1(c_0, 1-h_0)(1-\delta)k_0)}_{W(c_0,h_0,k_0,\lambda)} \\
&\quad + \sum_{t=1}^{\infty} \beta^t \underbrace{[u(c_t, 1-h_t) - \lambda u_1(c_t, 1-h_t)c_t - \lambda u_2(c_t, 1-h_t)h_t]}_{V(c_t,h_t,\lambda)} + \text{[MC], [CA] terms} \\
&= W(c_0, h_0, k_0, \lambda) + \sum_{t=1}^{\infty} V(c_t, h_t, \lambda) + \text{[MC], [CA] terms}
\end{aligned}$$

the lagrangian became:

$$\hat{V}(\{c_t, h_t\}_{t=0}^{\infty}) = W(c_0, h_0, k_0, \lambda) + \sum_{t=1}^{\infty} V(c_t, h_t, \lambda) + \text{[MC], [CA] terms}$$

and with this function we can formulate an equivalent problem:

$[RP3]$:

$$\max_{c_t,x_t,k_{t+1},h_t} \hat{V}(\{c_t, h_t, \lambda\}_{t=0}^{\infty})$$

$$\text{s.t.} \qquad [MC], [CA] \quad \forall t$$

Equilibrium conditions:

$$\begin{aligned}
\frac{V_2(t)}{V_1(t)} &= -F_2 \quad \text{for } t \geq 1 && \text{[RP1]}\\
\frac{V_1(t+1)}{V_1(t)} &= \beta(1-\delta+F_1) \quad \text{for } t \geq 1 && \text{[RP2]}\\
c_t + x_t + g_t &= F(k_t, h_t) && \text{[MC]}\\
k_{t+1} &= (1-\delta)k_t + x_t && \text{[CA]}
\end{aligned}$$

Note that:

$V_1(c_t, h_t, \lambda) = u_1(c_t, 1-h_t) - \lambda(u_{11}(c_t, 1-h_t)c_t + u_1(c_t, 1-h_t) - u_{21}(c_t, 1-h_t)h_t)$
$V_2(c_t, h_t, \lambda) = -u_2(c_t, 1-h_t) - \lambda(-u_{12}(c_t, 1-h_t)c_t + u_{22}(c_t, 1-h_t)h_t - u_2(c_t, 1-h_t))$

Chamley-Judd result on optimal taxation

Assume we are in the SS: $c_t^{RP} \to c^{RP}, h_t^{RP} \to h^{RP}, k_t^{RP} \to k^{RP}, x_t^{RP} \to x^{RP}$ then in the limit [RP2]:

$$\cancelto{1}{\frac{V_1(t+1)}{V_1(t)}} = \beta(1-\delta+F_1)$$

$$\boxed{1 = \beta(F_1(k^{RP}, h^{RP}) + 1 - \delta)}$$

The Ramsey allocation is a TDCE allocation that supports (g, τ) so it should be consistent with [H2]:

$$u_1(c_t^{RP}, 1-h_t^{RP}) = \beta u_1(c_{t+1}^{RP}, 1-h_{t+1}^{RP})[(1-\tau_{kt+1}^{RP})F_2(k_{t+1}^{RP}, h_{t+1}^{RP}) + 1 - \delta]$$

which in the limit becomes:

$$\boxed{1 = \beta\left[(1 - \lim_{t\to\infty} \tau_{kt+1})F_1(k^{RP}, h^{RP}) + 1 - \delta\right]}$$

both conditions in the boxes ([RP2] and [H2] in the limit) are compatible (supporting the same allocation) only if:

$$\boxed{\lim_{t\to\infty} \tau_{kt+1} = 0}$$

What about labor?: The Ramsey problem equilibrium condition ([RP1]) and TDCE condition ([H1]) are not conflicting with $\tau_{ht} > 0$,

the [RP1] is:

$$\begin{aligned}
\frac{V_2(c_t, h_t, \lambda)}{V_1(c_t, h_t, \lambda)} &= -F_2(k_t, h_t)\\
\frac{-u_2(c_t, 1-h_t) - \lambda(-u_{12}(c_t, 1-h_t)c_t + u_{22}(c_t, 1-h_t)h_t - u_2(c_t, 1-h_t))}{u_1(c_t, 1-h_t) - \lambda(u_{11}(c_t, 1-h_t)c_t + u_1(c_t, 1-h_t) - u_{21}(c_t, 1-h_t)h_t)} &= -F_2(k_t, h_t)\\
\frac{u_2(c_t, 1-h_t) - \lambda(u_{12}(c_t, 1-h_t)c_t - u_{22}(c_t, 1-h_t)h_t + u_2(c_t, 1-h_t))}{u_1(c_t, 1-h_t) - \lambda(u_{11}(c_t, 1-h_t)c_t + u_1(c_t, 1-h_t) - u_{21}(c_t, 1-h_t)h_t)} &= F_2(k_t, h_t)
\end{aligned}$$

whereas the [H1] is:

$$\frac{u_2(c_t, 1-h_t)}{u_1(c_t, 1-h_t)} = (1-\tau_{ht})F_2(k_t, h_t)$$

For this two conditions to be compatible and support the same allocation we need $\tau_{ht} \to 0$ only if $\lambda = 0$, i.e., if the [IC] is slack. That is the case when τ_{k0} is enough to pay for all $\{g_t\}$. This is not tipically the case and we will have $\tau_{ht} > 0$ in the limit.

14.5 OLG models in the NGM in discrete time

The NGM allows to analyse setups with homogeneous agents. Jointly with productivity shocks and uncertainty the RBC model can be formulated. Relevant results have been concluded from such model, e.g., that the volatility of investement is way greater than that of consumption as well as other results in monetary policy analysis.

However with homogeneous agents is not possible to model lifecycle behaviour and social security frameworks. To do this we will use the Overlapping Generations Model (OLG) where agents are born in different periods and live a finite amount of time.

Here **markets are incomplete** so the results won't be efficient. By incomplete we mean that agent won't be able to transfer resources among each other but will be quite limited in that regard instead: **Agents die and then do not overlap among all of each other**. Then we will need government to intervene (the government would redistribute wealth and induce optimal savings behavior), the existence of money or some type of contracts.

Setup:

Endowments: $e = (e^y, e^o) = (1, 0)$ (the endowments are in terms of working time)

Initially old agent UMP:

$$\begin{aligned} &\max_{c_0^o} u(c_0^o) \\ \text{s.t.} \quad & c_0^o = s_{-1}R_0 \end{aligned}$$

the solution is straightforward: $c_0^o = s_{-1}R_0$

Agents born at $t \geq 0$ UMP:

$$\begin{aligned} &\max_{c_t^y, c_{t+1}^o} u(c_t^y) + \beta u(c_{t+1}^0) \\ \text{s.t.} \quad & c_t^y + s_t = w_t && [BC^y] \\ & c_{t+1}^o = R_{t+1}s_t && [BC^o] \end{aligned}$$

Lifetime BC: $c_t^y + \frac{1}{R_{t+1}}c_{t+1}^o = w_t$

FOC:

$$\begin{aligned} \frac{u'(c_t^y)}{u'(c_{t+1}^o)} &= \beta R_{t+1} && \text{[EE]} \\ c_t^y + s_t &= w_t && [BC^y] \\ c_{t+1}^o &= R_{t+1}s_t && [BC^o] \end{aligned}$$

Population dynamics: $\frac{N_{t+1}}{N_t} = 1 + n$

Firms: $F(K,N)$ where i. $F(\cdot)$ is HOD1, ii. $F(0,N) = F(K,0) = 0$, iii. $F_K > 0, F_N > 0$, iv. $F_{KK} < 0.F_{NN} < 0$, v. $\lim\limits_{K \to 0} F(K,N) = 0$, $\lim\limits_{N \to 0} F(K,N) = 0$

with a HOD1 production function we will have HOD0 partial derivatives as well as zero profit conditions (given by Euler Theorem):

$$F(K,N) - wN - rK = 0$$

with

$$r = F_1(K,N) = F_1(k,1) = f'(k)$$

then in per-capita terms:

$$r = f'(k) \tag{F1}$$

$$w = f(k) - f'(k)k \tag{F2}$$

Market clearing: for the goods market we aggregate over agents considering the size of population of each age at a given t,

$$\begin{gathered} c_t^y N_t + c_t^o N_{t-1} + I_t = F(K_t, K_t) \\ c_t + \frac{c_t^0}{1+n} + i_t = f(k_t) \end{gathered}$$

the law of motion of capital and its per-capita form is:

$$\begin{aligned} K_{t+1} &= I_t + (1-\delta)K_t \\ k_{t+1}(1+n) &= i_t + (1-\delta)k_t \quad \Rightarrow \quad i_t = k_{t+1}(1+n) - (1-\delta)k_t \end{aligned}$$

substitute in the market clearing condition above,

$$c_t + \frac{c_t^0}{1+n} + \underbrace{k_{t+1}(1+n) - (1-\delta)k_t}_{i_t} = f(k_t)$$

$$c_t + \frac{c_t^0}{1+n} + k_{t+1}(1+n) = f(k_t) + (1-\delta)k_t \tag{MC}$$

On the other hand the asset market clearing condition is:

$$K_{t+1} = N_t s_t$$

in per-capita terms:

$$(1+n)k_{t+1} = s_t \tag{AM}$$

finally the no arbitrage condition in assets markets implies that in equilibrium:

$$\underbrace{R_t}_{\text{Gross returns in assets}} = \underbrace{r(t)+1-\delta}_{\text{Gross returns in capital}} \qquad \text{[NA]}$$

Solving the model:

Use the EE, replace the consumption in both periods ($c_t = w_t - s_t$, $c_{t+1} = R_{t+1}s_t$):

$$\frac{u'(w_t - s_t)}{u'(R_{t+1}s_t)} = \beta R_{t+1}$$

with this we can solve for a savings function: $s(R(k_{t+1}), w(k_t))$ and using [AM] we get:

$$k_{t+1} = \frac{s_t}{(1+n)} = \frac{s(R(k_{t+1}), w(k_t))}{1+n}$$

from here we solve for $k_{t+1} = f(k_t)$ (remember factor prices are set in terms of k_t) getting a **transition equation**.

Example: $u(c) = \ln c$ and $F(K,N) = AK^{\alpha}N^{1-\alpha}$

$$\begin{aligned}
\frac{u'(c_t)}{u'(c_{t+1})} &= \beta R_{t+1}\\
\frac{c_{t+1}}{c_t} &= \beta R_{t+1}\\
\frac{R_{t+1}s_t}{w_t - s_t} &= \beta R_{t+1}\\
s_t &= \frac{\beta}{1+\beta}w_t
\end{aligned}$$

using [AM]:

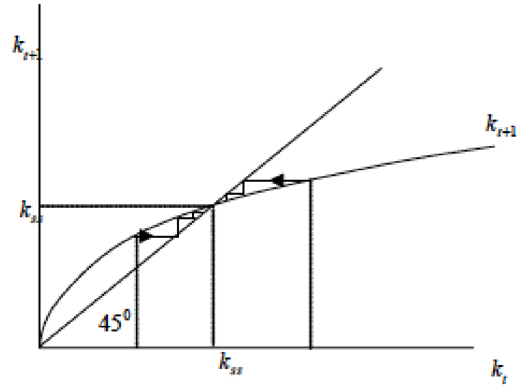
$$\begin{aligned}
k_{t+1} &= \frac{s_t}{1+n}\\
&= \frac{\beta}{1+\beta}\frac{w_t}{1+n}\\
&= \frac{\beta}{(1+\beta)(1+n)}(f(k_t) - f'(k_t)k_t)\\
&= \frac{\beta}{(1+\beta)(1+n)}(Ak_t^{\alpha} - \alpha Ak_t^{\alpha})
\end{aligned}$$

then:

$$\boxed{k_{t+1} = \tfrac{A\beta(1-\alpha)}{(1+\beta)(1+n)}k_t^{\alpha}}$$

and in SS:

$$k_{ss} = \left[\frac{A\beta(1-\alpha)}{(1+\beta)(1+n)}\right]^{\frac{1}{1-\alpha}}$$



in the graph is depicted both the function and the SS as a fixed point.

finally with k_t all other variables are found:

$$\begin{aligned}
&w_t = f(k_t) - f'(k_t)k_t = A(1-\alpha)k_t^\alpha\\
&r_t = f'(k_t) = \alpha A k_t^{\alpha-1}\\
&R_t = r_t + 1 - \delta = \alpha A k_t^{\alpha-1} + 1 - \delta\\
&s_t = \tfrac{\beta}{1-\beta} w_t = \tfrac{\beta}{1-\beta}(f(k_t) - f'(k_t)k_t) = \tfrac{A\beta(1-\alpha)}{(1+\beta)} k_t^\alpha\\
&c_t^y = w_t - s_t = w_t - \tfrac{\beta}{1+\beta} w_t = \tfrac{1}{1+\beta} A(1-\alpha)k_t^\alpha\\
&c_{t+1}^o = s_t R_{t+1} = \tfrac{A\beta(1-\alpha)}{(1+\beta)} k_t^\alpha \left(\alpha A k_{t+1}^{\alpha-1} + 1 - \delta\right) = \tfrac{A\beta(1-\alpha)}{(1+\beta)} k_t^\alpha \left(\alpha A \left[\tfrac{A\beta(1-\alpha)}{(1+\beta)(1+n)} k_t^\alpha\right]^{\alpha-1} + 1 - \delta\right)
\end{aligned}$$

Summary

OLG eq. conditions

$$\begin{aligned}
\frac{u'(c_t^y)}{u'(c_{t+1}^o)} &= \beta R_{t+1} && \text{[EE]}\\
c_t^y + s_t &= w_t && [BC]^y\\
c_{t+1}^o &= s_t R_{t+1} && [BC]^o\\
r_t &= f'(k_t) && [F1]\\
w_t &= f(k_t) - k_t f'(k_t) && [F2]\\
R_t &= r_t + 1 - \delta && [NA]\\
c_t^y + \frac{c_t^o}{1+n} + (1+n)k_{t+1} &= f(k_t) + (1-\delta)k_t && [MC]\\
(1+n)k_{t+1} &= s_t && [AM]
\end{aligned}$$

Dynamic inefficiency

An economy is dynamically efficient if no agent can be made better off without making someone else worse off.

Proposition: The OLG economy is dynamically inefficient iff

$$f'(k_{ss}) < n + \delta$$

Proof:

'$\Leftarrow$' (assume $f'(k_{ss}) < n+\delta$)

take que market clearing condition in the SS:

$$c^y_{ss} + \frac{c^o_{ss}}{1+n} + (1+n)k_{ss} - (1-\delta)k_{ss} = f(k_{ss}) \qquad [MC]^{ss}$$

solve for the aggregate consumption and optimize:

$$c^y_{ss} + \frac{c^o_{ss}}{1+n} = f(k_{ss}) - (n+\delta)k_{ss}$$

FOC:

$[k_{ss}]: \quad f'(k_{ss}) - (n+\delta) = 0 \quad \Rightarrow \quad \boxed{f'(k_{ss}) = n+\delta}$ ("efficient" level of capital to optimize consumption)

but our assumption is that $f'(k_{ss}) < n+\delta$ and the production function is strictly concave which means that our assumed capital level is dynamically inefficient, it is greater than the efficient level.

$\hookrightarrow$

This completes one direction of the proof. However we can go further and build a Pareto Improving allocation in such case: Since, $k_{ss} = \frac{s_{ss}}{1+n}$, we need to decrease the savings as well. To see that denote the improved allocation by k^{PI}_{ss} and define it such that:

$$f'(k_{ss}) < f'(k^{PI}_{ss}) < (n+\delta)$$

then

$$f(k^{PI}_{ss}) - (n+\delta)k^{PI}_{ss} > f(k_{ss}) - (n+\delta)k_{ss}$$

which means that aggregate consumption has increased accordingly, let the new levels of consumption (achieved by increasing youngsters consumption by ε_1) be: $\bar{c}^y_{ss} = c^y_{ss} + \varepsilon_1$ and $\bar{c}^o_{ss} = c^o_{ss} - (1+n)\varepsilon_2$

Here $\varepsilon_2 < \varepsilon_1$ which reflects the fact that lowering k_{ss} lowers s_{ss} proportionally but w_{ss} less than proportionally.

Now, its clear that in the new allocation $\{\bar{c}^y_{ss}, \bar{c}^o_{ss}\}$ the olders are worse off, then implement a transfer scheme given by $\varepsilon_2 < \tau < \varepsilon_1$ that improves both consumptions with respect to the initial allocation: $c^{y,PI}_{ss} = c^y_{ss} + \varepsilon_1 - \tau$ and $c^{o,PI}_{ss} = c^o_{ss} - (1+n)\varepsilon_2 + \tau(1+n)$

$\hookleftarrow$

'$\Rightarrow$' (Dyn. Inefficient $\quad \Rightarrow \quad f'(k_{ss}) < (n+\delta)$)

This is analog to prove the contrapositive: $f'(k_{ss}) > (n+\delta) \quad \Rightarrow \quad$ Dynamically efficient

(by contradiction) Suppose not, then there is a way to apply a pareto improvement. To get the efficiency condition k_{ss} should be increased, i.e., we need to increase savings s_{ss}. When doing so we need to decrease consumption of the young people $\downarrow c^y_{ss}$, to compensate we would require a transfer from the old, but it makes them worse off, therefore it cannot be Pareto Improving (We have concluded that if $f'(k_{ss}) > n+\delta$ then the allocation is Dynacmically Efficient).

Main intuition: In this case to get to the optimun we need to increase the capital, i.e., hurt someone's consumption.

Condition for Dynamic Efficiency

in SS:

$$k_{ss} = \left[\frac{A\beta(1-\alpha)}{(1+\beta)(1+n)}\right]^{\frac{1}{1-\alpha}}$$
$$f'(k_{ss}) = A\alpha k_{ss}^{\alpha-1}$$

subs. these results in the dynamic inefficiency condition:

$$\begin{aligned} f'(k_{ss}) &< n+\delta \\ A\alpha k_{ss}^{\alpha-1} &< n+\delta \\ A\alpha\left[\frac{A\beta(1-\alpha)}{(1+\beta)(1+n)}\right]^{\frac{\alpha-1}{1-\alpha}} &< n+\delta \\ \frac{\alpha(1+\beta)(1+n)}{\beta(1-\alpha)} &< n+\delta \end{aligned}$$

Example: Data check

Suppose we look at the data and find annual population growth of 1%, $\frac{rk}{y} = .3$, annual depreciation equal to 15%. Assume the length of the period is 30 years. For which values of β the model is dynamically efficient?

first we can get the relevant parameters:

$$\begin{aligned} (1+n) &= 1.01^{30} = 1.35 \\ \delta &= 1-(1-.15)^{30} \approx 1 \\ \alpha &= .3 \end{aligned}$$

dynamic inefficiency occurs if:

$$\begin{aligned} \frac{\alpha(1+\beta)(1+n)}{\beta(1-\alpha)} &< n+\delta \\ \frac{.3(1+\beta)(1.35)}{\beta(1-.3)} &< 1.35 \\ .3(1+\beta) &< .7\beta \\ \beta &> .75 \end{aligned}$$

Example: Let $A = 1, \alpha = .3, \delta = 1, n = .35, \beta = .9$, show that it is possible to construct a welfare improving scheme.

Solve for the SS:

$k_{ss} = \left[\frac{A\beta(1-\alpha)}{(1-\beta)(1+n)}\right]^{\frac{1}{1-\alpha}} = \left[\frac{.9(1-.3)}{(1+.9)(1+.35)}\right]^{\frac{1}{1-.3}} = .1346$
$w_{ss} = A(1-\alpha)k_{ss}^{\alpha} = (1-.3)(.1346)^{.3} = .38354$
$s_{ss} = \frac{\beta}{1+\beta}w_t = \frac{.9}{1+.9}(.38354) = 0.18168$
$c_{ss}^{y} = \frac{\beta}{1+\beta}w_{ss} = \frac{1}{1+.9}(.38354) = 0.20186$

$c^o_{ss} = [Ak^{\alpha}_{ss} - (n+\delta)k_{ss} - c^y_{ss}](1+n) = [.1346^{.3} - 1.35(.1346) - .20186]1.35 = 0.22187$

Now we solve for the level of capital that maximizes consumption per worker:

$$\begin{aligned} f'(k^*_{ss}) &= n+\delta \\ \alpha A k^{*\alpha-1}_{ss} &= n+\delta \\ k^*_{ss} &= \left(\frac{n+\delta}{\alpha A}\right)^{\frac{1}{\alpha-1}} = \left(\frac{1.35}{.3}\right)^{\frac{1}{.3-1}} = 0.11664 \end{aligned}$$

the level of savings to mantain that amount of capital is:

$$s^*_{ss} = (1+n)k^*_{ss} = 1.35(0.11664) = 0.15746$$

As expected that lower of savings is lower than the steady state level:

$$s_{ss} > s^*_{ss}$$
$$0.18168 > 0.15746$$

i.e., the young are oversaving, then **by setting any save rate between $[s^*_{ss}, s_{ss}]$ we can get a welfare improving scheme**.

To find a SP allocation

1. find k^*_{ss} s.t. $f'(k^*_{ss}) = n+\delta$
2. using k^*_{ss} and MC, solve for $\frac{c^{o*}_{ss}}{1+n}$,

$$\frac{c^o_{ss}}{1+n} = f(k^*_{ss}) - (n+\delta)k^*_{ss} - c^{y*}_{ss}$$

3. subs. $\frac{c^{o*}_{ss}}{(1+n)}$ in the EE: $\frac{\beta c^{y*}_{ss}}{c^{o*}_{ss}}$ and find c^{y*}_{ss}:

$$\beta c^{y*}_{ss} = \frac{c^{o*}_{ss}}{1+n} = f(k^*_{ss}) - (n+\delta)k^*_{ss} - c^{y*}_{ss}$$

$$c^*_{ss} = \frac{f(k^*_{ss}) - (n+\delta)k^*_{ss}}{1+\beta}$$

4. find $c^{o*}_{ss} = (f(k^*_{ss}) - (n+\delta)k^*_{ss} - c^{y*}_{ss})(1+n)$

To find the optimal τ^*

1. use k^*_{ss} to find w^*_{ss} and s^*_{ss}:

$$w^*_{ss} = f(k^*_{ss}) - f'(k^*_{ss})k^*_{ss}$$
$$s^*_{ss} = k^*_{ss}(1+n)$$

2. find: $c^{y*}_{ss} = w^*_{ss} - s^*_{ss}$
3. find: $\tau^* = \bar{c}^y_{ss} - c^{y*}_{ss}$ where $\bar{c}^y_{ss}$ is the consumption at the golden rule, i.e. consistent with $f'(k_{ss}) = n+\delta$

The τ equation shows that the optimal transfer amounts to the difference between the optimal consumption and that obtained as solution in the steady state.

14.5.1 Social Security

Fully funded (FF)

$$\max_{c_t^y, c_{t+1}^o} u(c_t^y) + \beta u(c_{t+1}^0)$$

$$\text{s.t.} \quad c_t^y + s_t + \mathbf{b_t} = w_t \qquad [BC^y]$$

$$c_{t+1}^o = R_{t+1} s_t + \mathbf{b_t R_{t+1}} \qquad [BC^o]$$

Lifetime BC: $c_t^y + \frac{1}{R_{t+1}} c_{t+1}^o = w_t$

with log-utility:

$$\frac{(s_t + b_t) R_{t+1}}{w_t - (s_t + b_t)} = \beta R_{t+1} \qquad \text{[EE]}$$

then:

$$s_t + b_t = \frac{\beta}{1+\beta} w_t$$

Cases:

- $b_t \leq \frac{\beta}{1+\beta} w_t \quad \Rightarrow \quad$ total savings: $s_t + b_t = \frac{\beta}{1+\beta} w_t$
then $k_{t+1} = \frac{b_t + s_t}{(1+n)} = \frac{\beta}{(1+\beta)(1+n)} w_t = \frac{\beta}{(1+\beta)(1+n)} A k_t^\alpha$ (not distorting)

- $b_t \leq \frac{\beta}{1+\beta} w_t \quad \Rightarrow \quad$ over acummulation of capital and $\mathbf{s_t = 0}$ (least possible distorting savings)
then $k_{t+1} = \frac{b_t}{(1+n)} > \frac{\beta}{(1+\beta)(1+n)} w_t$ (distorting savings, non efficient outcome)

Pay as you go (PAYGO)

$$\max_{c_t^y, c_{t+1}^o} u(c_t^y) + \beta u(c_{t+1}^0)$$

$$\text{s.t.} \quad c_t^y + s_t = w_t - \boldsymbol{\tau} \qquad [BC^y]$$

$$c_{t+1}^o = R_{t+1} s_t + (\mathbf{1 + n}) \tau \qquad [BC^o]$$

Lifetime BC: $c_t^y + \frac{1}{R_{t+1}} c_{t+1}^o = w_t$

with log-utility:

$$\frac{c_{t+1}^o}{c_t^y} = \beta R_{t+1} \qquad \text{[EE]}$$

then:

$$\frac{s_t R_{t+1} + \tau(1+n)}{w_t - \tau - s_t} = \beta R_{t+1}$$

$$\frac{\beta(w_t - \tau)}{1+\beta} - \tau \frac{(1+n)}{(1+\beta) R_{t+1}} = s_t$$

subs. w_t, $R_{t+1} = 1 + r_{t+1} - \delta$ and use $k_{t+1} = \frac{s_t}{1+n}$

$$k_{t+1} = \frac{\beta(A(1-\alpha)k_t^{\alpha} - \tau)}{(1+\beta)(1+n)} - \frac{\tau}{(1+\beta)(1+\alpha A k_{t+1}^{\alpha-1}) - \delta}$$

let $\delta = 1$,

$$k_{t+1} = \frac{\beta(A(1-\alpha)k_t^{\alpha} - \tau)}{(1+\beta)(1+n)} - \frac{\tau}{(1+\beta)(\alpha A k_{t+1}^{\alpha-1})}$$

define $\Phi(k_{ss}, \tau) = k_{t+1} - \frac{\beta(A(1-\alpha)k_t^{\alpha} - \tau)}{(1+\beta)(1+n)} - \frac{\tau}{(1+\beta)(\alpha A k_{t+1}^{\alpha-1})} = 0$

use implicit function theorem:

$$\frac{dk_{ss}}{d\tau} = -\frac{\partial\Phi/\partial\tau}{\partial\Phi/\partial k_{ss}} = -\frac{\frac{1}{1+\beta}\left(\frac{\beta}{1+\beta} + \frac{1}{\alpha A k_{ss}^{\alpha-1}}\right)}{1 - \frac{1}{1+\beta}\left(\frac{\beta\alpha(1-\alpha)Ak_{ss}^{\alpha-1}}{1+n} - \frac{(1-\alpha)\tau}{\alpha k_{ss}^{\alpha}}\right)}$$

both numerator and denominator are positive, then $\boxed{\frac{\partial k_{ss}}{\partial \tau} < 0}$

Conclusion: FF system is at best ineffective (does nothing or distorts) whereas in the PAYGO welfare is increased in $a + n > R$ or decrease welfare otherwise.

However in reality n is relatively low and therefore many countries face problems of sustainability.Hence FF would be better now but is very costly to implement.

Summary

Assume an endowment of (y,y') during both periods of life, the wealth in each scheme is:

$$we^{No \quad SS} = y + \frac{y'}{1+r}, \qquad we^{PAYGO} = y + \frac{y'}{1+r} - b + b\frac{(1+n)}{(1+r)}, \qquad we^{FF} = y + \frac{y'}{1+r}$$

Notice that whenever $n = r$ the welfare and wealth is the same in each scheme. Then we can state that: the economy with no SS is **dynamically inefficient** when $\boxed{n > r}$, i.e., a PI outcome is reachable by implementing a PAYGO system.

$$we^{PAYGO} > we^{No \ SS} \quad if \quad b\frac{1+n}{1+r} > b \quad if \quad n > r$$

in such case the gov. can intervene an economy w/o SS and improve every agent.

Social security reform (example)

- to simplify assume there is no production.
- endownments: wealth = (y, y') = (50,0)
- Initial state: PAYGO system with transfer b also:

$$\begin{aligned} 1 + n &= 1 \\ R &= 1.05 \\ b &= 20 \\ \beta &= 1 \end{aligned}$$

Under PAYGO:

$$\max_{c,c'} \quad lnc + \beta lnc'$$

s.t.

$$c + s = y - b$$

$$c' = y' + sR + (1+n)b$$

Eq. conds.:

$$\frac{c'}{c} = \beta R \quad \Rightarrow \quad \frac{c'}{c} = 1.05$$

$$c + \frac{c'}{R} = y + \frac{y'}{R} - b + \frac{(1+n)}{R}b \quad \Rightarrow \quad c + \frac{c'}{1.05} = 50 - 20 + 20/1.05 = 49.04$$

then $c^* = 24.52$, $c'^* = 25.75$ and $u(c^*, c'^*)_{PAYGO} = 6.44$

Under FF:

$$\max_{c,c'} \quad lnc + \beta lnc'$$

s.t.

$$c + s = y - b$$

$$c' = y' + (s+b)R$$

Eq. conds.:

$$\frac{c'}{c} = \beta R \quad \Rightarrow \quad \frac{c'}{c} = 1.05$$

$$c + \frac{c'}{R} = y + \frac{y'}{R}$$

then $c^* = 25$, $c'^* = 26.25$ and $u(c^*, c'^*)_{FF} = 6.5$

also solution is interior: $b <$total savings w/o SS: $s + b = y - c = 50 - 25 = 25$ then the agent pays 20 to the government and saves 5 when SS is imposed.

in this case the economy is better off under a FF system given that $n < r$, if possible one would like to change from PAYGO to FF.

The problem is that, to implement such change the government would have to run a defficit (issue bonds) to pay for the benefits of the agents who are old at the time of the switch and pay that off by taxing future generations that benefit from the change. The problem is that it would take infinitely many generations to pay for the reform, i.e., the reform cannot pay for itself.

Reform:

-Suppose the switch ocurrs at period $\bar{T}$

- there are $\bar{N}$ old people, hence the government should pay then $\mathbb{N}b(1+n)$

- the gov. levies a tax t to all future generations to pay for the change

The present value of the tax revenues to pay for the reform is given by the RHS of the following expression (the LHS just states what it should be equal to, i.e., the total cost of the reform):

$$\bar{N}b(1+n) = \overbrace{t\bar{N}(1+n)}^{\#\text{ of young at }\bar{T}} + t\bar{N}\frac{(1+n)^2}{(1+r)} + t\bar{N}\frac{(1+n)^3}{(1+r)^2} + \cdots + t\bar{N}\frac{(1+n)^x}{(1+r)^{x-1}}$$

here x denotes the last generation to be taxed.

Now find the tax that extracts all the benefit associated with the reform, that is, t such that $Weath^{FF} = Wealth^{PAYGO}$

$$\begin{aligned} y + \frac{y'}{1+r} - \mathbf{t} &= y - b + \frac{y'}{1+r} + \frac{b(1+n)}{1+r} \\ -t &= -b + \frac{b(1+n)}{1+r} \\ t &= \frac{b(1+n)}{1+r} = \frac{20(0.05)}{1.05} = 0.95 \end{aligned}$$

Now, use t the government BC above to see how many generations should be taxed (removed the benefit from the change) to pay for the reform. Notice that the government won't provide more transfers after the reform, it only will tax what's required to implement the change:

$$\begin{aligned} \bar{N}b(1+n) &= \bar{N}\left[t(1+n) + \frac{t(1+n)^2}{(1+r)} + \frac{t(1+n)^2}{(1+r)} + \frac{t(1+n)^3}{(1+r)^2} + \cdots + \frac{t(1+n)^x}{(1+r)^{x-1}}\right] \\ 20 &\approx .95 + \left(\frac{1}{1.05}\right) + \left(\frac{1}{1.05}\right)^2 + \left(\frac{1}{1.05}\right)^3 + \cdots + \left(\frac{1}{1.05}\right)^\infty \end{aligned}$$

More generally we can show that the present value of the tax revenues amounts exactly to the cost of the reform.

To do that find the present value and replace for the tax found above: $t = \frac{b(r-n)}{1+r}$:

$$\begin{aligned} \bar{N}t(1+n) + \frac{\bar{N}t(1+n)^2}{1+r} + \frac{\bar{N}t(1+n)^3}{(1+r)^2} + \cdots &= \bar{N}t(1+n)\left[1 + \frac{(1+n)}{(1+r)} + \frac{(1+n)^2}{(1+r)^2} + \cdots\right] \\ &= \bar{N}t(1+n)\frac{1}{1-\frac{(1+n)}{(1+r)}} \\ &= \bar{N}\frac{b(r-n)}{1+r}(1+n)\frac{1}{\frac{r-n}{1+r}} \\ &= \bar{N}b(1+n) \end{aligned}$$

14.5.2 OLG with exogenous growth

In general is needed to incorporate exogenous growth (labor augmenting tech progress) in the production function:

$$F(K_t, N_t) = AK_t^\alpha((1+g)^t N_t)^{1-\alpha}$$

then

$$f(k_t) = A(1+g)^{(1-\alpha)t}k_t^\alpha$$

the eq. conditions were:

$$\frac{u'(c_t^y)}{u'(c_{t+1}^o)} = \beta R_{t+1} \tag{EE}$$

$$c_t^y + s_t = w_t \tag{$[BC]^y$}$$

$$c_{t+1}^o = s_t R_{t+1} \tag{$[BC]^o$}$$

$$r_t = f'(k_t) \tag{$[F1]$}$$

$$w_t = f(k_t) - k_t f'(k_t) \tag{$[F2]$}$$

$$R_t = r_t + 1 - \delta \tag{$[NA]$}$$

$$c_t^y + \frac{c_t^o}{1+n} + (1+n)k_{t+1} = f(k_t) + (1-\delta)k_t \tag{$[MC]$}$$

$$(1+n)k_{t+1} = s_t \tag{$[AM]$}$$

and in terms of the log-utility and new production function become:

$$\frac{c_{t+1}^o}{c_t^y} = \beta R_{t+1} \tag{EE}$$

$$c_t^y + s_t = w_t \tag{$[BC]^y$}$$

$$c_{t+1}^o = s_t R_{t+1} \tag{$[BC]^o$}$$

$$r_t = A\alpha(1+g)^{(1-\alpha)t} k_t^{\alpha-1} \tag{$[F1]$}$$

$$w_t = A(1-\alpha)(1+g)^{(1-\alpha)t} k_t^{\alpha} \tag{$[F2]$}$$

$$R_t = r_t + 1 - \delta \tag{$[NA]$}$$

$$c_t^y + \frac{c_t^o}{1+n} + (1+n)k_{t+1} = A(1+g)^{(1-\alpha)t} k_t^{\alpha} + (1-\delta)k_t \tag{$[MC]$}$$

$$(1+n)k_{t+1} = s_t \tag{$[AM]$}$$

to solve for the equilibrium by using efficiency variables:

$$\begin{aligned}
\hat{c}_t^y &= \frac{c_t^y}{(1+g)^t} \\
\hat{c}_t^o &= \frac{c_t^o}{(1+g)^t} \\
\hat{k}_t &= \frac{k_t}{(1+g)^t} \\
\hat{s}_t &= \frac{s_t}{(1+g)^t} \\
\hat{w}_t &= \frac{w_t}{(1+g)^t} \\
\hat{r}_t &= r_t \\
\hat{R}_t &= R_t
\end{aligned}$$

the FOC rewritten in terms of efficiency variables are:

$$\frac{(1+g)c^o_{t+1}}{c^y_t} = \beta \hat{R}_{t+1} \tag*{[EE]}$$
$$\hat{c}^y_t + \hat{s}_t = \hat{w}_t \tag*{$[BC]^y$}$$
$$(1+g)c^o_{t+1} = \hat{s}_t \hat{R}_{t+1} \tag*{$[BC]^o$}$$
$$\hat{r}_t = A\alpha \hat{k}_t^{\alpha-1} \tag*{$[F1]$}$$
$$\hat{w}_t = A(1-\alpha)\hat{k}_t^{\alpha} \tag*{$[F2]$}$$
$$\hat{R}_t = r_t + 1 - \delta \tag*{$[NA]$}$$
$$\hat{c}^y_t + \frac{\hat{c}^o_t}{1+n} + (1+n)(1+g)\hat{k}_{t+1} = A\hat{k}_t^{\alpha} + (1-\delta)\hat{k}_t \tag*{$[MC]$}$$
$$(1+n)(1+g)\hat{k}_{t+1} = \hat{s}_t \tag*{$[AM]$}$$

we follow the standard procedure: replace the consumptions in the EE and solve for the savings, then pin down the transition equation (using the AM eq.) and the level of capital in the SS:

$$\frac{\hat{s}_t \hat{R}_{t+1}}{\hat{w}_t - \hat{s}_t} = \beta \hat{R}_{t+1} \quad \Rightarrow \quad s_t = \frac{\beta}{1+\beta} w_t$$
$$\hat{k}_{t+1} = \frac{s_t}{(1+n)(1+g)} = \frac{\beta}{(1+\beta)(1+n)(1+g)}\hat{w}_t \quad \Rightarrow \quad k_{t+1} = \frac{(1-\alpha)A\beta}{(1+\beta)(1+n)(1+g)}\hat{k}_t^{\alpha}$$

and therefore

$$\boxed{\hat{k}_{ss} = \left[\frac{(1-\alpha)A\beta}{(1+\beta)(1+n)(1+g)}\right]^{\frac{1}{1-\alpha}}}$$

the economy will be Dynamically inefficient if:

$$\hat{r}_{ss} = \alpha A \hat{k}_{ss}^{\alpha-1} < \delta + g + n + ng$$
$$\alpha A \left[\frac{(1-\alpha)A\beta}{(1+\beta)(1+n)(1+g)}\right]^{\frac{\alpha-1}{1-\alpha}} < \delta + g + n + ng$$
$$\frac{\alpha(1+\beta)(1+n)(1+g)}{(1-\alpha)\beta} < \delta + g + n + ng$$

Example: (calibration)Is the postwar U.S. economy dynamically efficient? use the data parameters considered before: quarterly pop. growth is 0.35%, quarterly growth rate of real output per worker is 0.46%, capital share in total income is 0.37 and quarterly capital-output ratio is 12.56 (12.56/4 = 3.14 in annual terms) and annual depreciation rate is 8%

$$1 + n = 1.0035^{(4)35} = 1.63$$
$$1 + g = 1.0046^{(4)35} = 1.9$$
$$\alpha = .37$$
$$1 - \delta = (1 - .08)^{35} \quad \Rightarrow \quad \delta = 0.946$$

then the value of β for which the U.S. economy is dynamically inefficient is:

$$\begin{aligned}\frac{\alpha(1+\beta)(1+n)(1+g)}{(1-\alpha)\beta} &< \delta+g+n+ng\\ \frac{.37(1+\beta)(1.63)(1.9)}{(1-.37)\beta} &< .946+.9+.63+.63(.9)\\ 1.1459(1+\beta) &< \beta\\ 1.486 &< \beta\end{aligned}$$

then for usual values of β the economy is dynamically efficient.

15 NGM with Uncertainty

Main results:

- The risk will be perfectly shared.
- The distribution of endowments across households is irrelevant.

Setup:

- states: $s_t \in S : \{\eta_1, \cdots, \eta_N\}$ where S is the space of possible events, here we will use the notation: $s_t = 1$ instead of $s_t = \eta_1$
- s^t: event history up to period t, i.e., $s^t = (s_0, s_1, s_2, \cdots, s_t)$
- Also $\pi_t(s^t) > 0$ is the likelihood of observing the history s^t
- Goods have two indexes: time and state (history), then we have to sum over both time and histories in the BC.
- Suppose two agents, an allocation is given by: $(c^1, c^2) = \{c_t^1(s^t), c_t^2(s^t)\}_{t=0,s^t\in S^t}^{\infty}$, and the endowments are: $(e^1, e^2) = \{e_t^1(s^t), e_t^2(s^t)\}_{t=0,s^t\in S^t}^{\infty}$
- Agents max. the expected utility: $u(c^i) = \sum_{t=0}^{\infty}\sum_{s^t\in S^t}\beta^t\pi_t(s^t)U(c_t^i(s^t))$
- We normalize by one commodity: $p_0(s_0 = 1) = 1$

Competitive equilibrium - AD

prices $\{\hat{p}_t(s^t)\}_{t=0,s^t\in S^t}^{\infty}$ and allocations $\{\hat{c}_t^i(s^t)\}_{t=0,s^t\in S^t}^{\infty}$ such that, for all i:

1. Given prices, c^i solves the UMP:

$$\max u(c^i) = \sum_{t=0}^{\infty}\sum_{s^t\in S^t}\beta^t\pi_t(s^t)U(c_t^i(s^t))$$

$$\text{s.t.}\quad \sum_{t=0}^{\infty}\sum_{s^t\in S^t}\hat{p}_t(s^t)\hat{c}_t^i(s^t) \le \sum_{t=0}^{\infty}\sum_{s^t\in S^t}\hat{p}_t(s^t)\hat{e}_t^i(s^t)$$

$$\hat{c}_t^i \ge 0$$

2. MC:

$$c_t^1(s^t) + c_t^2(s^t) = e_t^1(s^t) + e_t^2(s^t)$$

Notice that there are not probabilities in the BC, this happens because they are already included in the prices.

FOC:

$$[c_t^i(s^t)] : \qquad \beta^t \pi_t(s^t) U'(c_t^i(s^t)) = \mu p_t(s^t) \quad \forall t$$

for t=0: $\qquad \pi_0(s^0) U'(c_0^i(s^0)) = \mu p_0(s^0)$

then:

$$\frac{p_t(s^t)}{p_0(s_0)} = \beta^t \frac{\pi_t(s^t)}{\pi_0(s_0)} \frac{U'(c_t^i(s^t))}{U'(c_0^i(s_0))}$$

this holds for every agent which means that **the ratio of marginal utilities between the two agents is constant over time:**

$$\frac{U'(c_t^1(s^t))}{U'(c_0^1(s_0))} = \frac{U'(c_t^2(s^t))}{U'(c_0^2(s_0))} \qquad \Rightarrow \qquad \frac{U'(c_t^2(s^t))}{U'(c_1^1(s^t))} = \frac{U'(c_0^2(s_0))}{U'(c_0^1(s_0))} \quad \forall s^t$$

e.g. with a CRRA: $\left(\frac{c_t^2(s^t)}{c_t^1(s^t)}\right)^{-\sigma} = \left(\frac{c_0^2(s_0)}{c_0^1(s_0)}\right)^{-\sigma} \quad \Rightarrow \quad \frac{c_t^2(s^t)}{c_t^1(s^t)} = \frac{c_0^2(s_0)}{c_0^1(s_0)} = constant$

define the aggregate endowments: $\quad e_t(s^t) = \sum_i e_t^i(s^t) \quad \Rightarrow \quad \boxed{c_t^i(s^t) = \phi^i e_t(s^t)}$

i.e. given that the consumption ratio among agents is constant then the share of aggregate endowments also is. Then endowment risk is perfectly shared since it only depends on aggregate endowments. **Shocks to individual or relative endowments do not affect consumption**.

let $p_0(s_0) = 1$, then:

$$\frac{p_t(s^t)}{\cancel{p_0(s_0)} \to 1} = \beta^t \frac{\pi_t(s^t)}{\pi_0(s^0)} \left(\frac{c_t^i(s^t)}{c_0^i(s_0)}\right)^{-\sigma}$$

$$\boxed{p_t(s^t) = \beta^t \frac{\pi_t(s^t)}{\pi_0(s^0)} \left(\frac{e_t(s^t)}{e_0(s_0)}\right)^{-\sigma}}$$

The prices equation reflect the time discounting (goods are less valuable in the future) and also the likelihood ratio of a given state: the higher the probabiltity of reaching a given state the higher the correspondent good is priced. Finally it reflects the relative availability of resources, the higher availability (measured by aggregate endowments) the lower the price.

Analogously, the prices depend on the stochastic process of aggregate endowment only and not in its distribution.

Pareto Efficiency

Feasibility: $(c^1, c^2) = \{c_t^1(s^t), c_t^2(s^t)\}_{t=0, s^t \in S^t}^{\infty}$ is feasible if for all i:

1. $c^i_t(s^t) \geq 0 \quad \forall t, s^t \in S^t$
2. $c^1_t(s^t) + c^2_t(s^t) = e^1_t(s^t) + e^2_t(s^t) \quad \forall t, s^t \in S^t$

Pareto Optimal: (c^1, c^2) is Pareto efficient if it is feasible and there is no other feasible allocation $(\tilde{c}^1, \tilde{c}^2) = \{\tilde{c}^1_t(s^t), \tilde{c}^2_t(s^t)\}^{\infty}_{t=0, s^t \in S^t}$ such that:

$$u(\tilde{c}^i) \geq u(c^i) \quad \forall i, \text{with at least one strict inequality}$$

First Welfare Theorem: $\{c^2_t(s^t)\}^{\infty}_{t=0,s^t \in S^t}$ is a CE allocation then it is Pareto Efficient.

Sequential Markets equilibrium - SM

- Before: w/o uncertaintey there was trade each period and it was used 1 period securities.

- Now: With risk, there are 1 period state contingent securities $q_t(s^t, s_{t+1} = j)$: price at t of contract that pays one unit of consumption in t+1, **if** $s_{t+1} = j$

- $a^i_{t+1}(s^t, s_{t+1})$: quantity of securities bought/sold at period t by agent i.

therefore the period t BC with history s^t is:

$$c^i_t(s^t) + \sum_{s_{t+1} \in S} q_t(s^t, s_{t+1}) a^i_{t+1}(s^t, s_{t+1}) \leq e^i_t(s^t) + a^i_t(s^t)$$

SE Equilibrium

Allocations $\{(\hat{c}^i_t(s^t), \{\hat{a}^i_{t+1}(s^t, s_{t+1})\}_{s_{t+1} \in S})\}^{\infty}_{t=0,s^t \in S^t}$ and prices $\{\hat{q}_t(s^t, s_{t+1})\}^{\infty}_{t=0,s^t \in S^t}$ such that for all i:

1. given prices the allocation $\{(\hat{c}^i_t(s^t), \{\hat{a}^i_{t+1}(s^t, s_{t+1})\}_{s_{t+1} \in S})\}^{\infty}_{t=0,s^t \in S^t}$ solves the UMP:

$$\max u(c^i) = \sum_{t=0}^{\infty} \sum_{s^t \in S^t} \beta^t \pi_t(s^t) U(c^i_t(s^t))$$

$$\text{s.t.} \qquad c^i_t(s^t) + \sum_{s_{t+1} \in S} q_t(s^t, s_{t+1}) a^i_{t+1}(s^t, s_{t+1}) \leq e^i_t(s^t) + a^i_t(s^t) \quad \forall t, s^t \in S^t$$

$$\hat{c}^i_t(s^t) \geq 0 \quad \forall t, s^t \in S^t$$

$$a^i_{t+1}(s^t, s_{t+1}) \geq -\bar{A}^i \quad \forall t, s^t \in S^t, s_{t+1} \in S$$

2. MC: for $t \geq 0$

$$\sum_{i=1}^{2} \hat{c}^i_t(s^t) \leq \sum_{i=1}^{2} e^i_t(s^t)$$

$$\sum_{i=1}^{2} \hat{a}^i_{t+1}(s^t, s_{t+1}) = 0$$

Equivalence between AD and SM

$$AD_{\text{w/ risk}} \equiv SM_{\text{w/ risk}}$$

As before to get equivalency it suffices to map the prices in the AD market with the inverse of the interest rates of the one period bonds in the SM structures making the relative prices in both frameworks to be the same.

the mapping between AD prices and securities prices would be:

$$q_t(s^t, s_{t+1}) = \frac{p_{t+1}(s^{t+1})}{p_t(s^t)}$$

where $p_t(s^t, s_{t+1}) = p_0(s_0) \times q_0(s_0, s_1) \times \cdots \times q_{t-1}(s^{t-1}, s_t)$

15.1 Asset Pricing

- we price with AD prices an asset j
- $d_j = \{d_t^i(s^t)\}$: dividends paid by asset j in each node s^t (claim of units of consumption)
- the time 0 price of asset j (cum dividend) is

$$P_0^j(d) = \sum_{t=0}^{\infty} \sum_{s^t} p_t(s^t) d_t^j(s^t)$$

this is the value of all consumption goods the asset delivers at all future dates and states.

- Ex-dividend price of such asset at node s^t expressed in terms of period t consumption (given history at t or at node (s^t)) goods is

$$P_t^j(d, s^t) = \frac{\sum_{\tau=t+1}^{\infty} \sum_{s^\tau | s^t} p_\tau(s^\tau) d_\tau^j(s^\tau)}{p_t(s^t)}$$

this price is just the cum dividend price for the periods that remain after t in terms of the prices already observed at the current node. Notice the numerator includes an expected value of future nodes s^τ conditional on being in node s^t, to simplify this will be modelled with a markov chain.

- One period gross real return of asset j between s^t and s^{t+1},

$$R_{t+1}^j(s^{t+1}) = \frac{P_{t+1}^j(d, s^t) + d_{t+1}^j(s^{t+1})}{P_t^j(d, s^t)}$$

Examples:

i. Asset bought in s^t that pays 1 in $\hat{s}_{t+1}$ and 0 in all other states s_{t+1}.

Price at s^t:

$$P_t^A(d, s^t) = \frac{p_{t+1}(\hat{s}^{t+1})}{p_t(s^t)} = q_t(s^t, \hat{s}_{t+1})$$

the gross return between s^t and $\hat{s}^{t+1} = (s^t, \hat{s}_{t+1})$ is:

$$R_t^A(\hat{s}^{t+1}) = \frac{0 + 1}{p_{t+1}(\hat{s}^{t+1})/p_t(s^t)} = \frac{p_t(s^t)}{p_{t+1}(\hat{s}^{t+1})} = \frac{1}{q_t(s^t, \hat{s}_{t+1})}$$

and $R_{t+1}^A(s^{t+1}) = 0$ for all $s_{t+1} \neq \hat{s}_{t+1}$

ii. One period risk free bond:

Price at s^t:

$$P_t^B(d, s^t) = \frac{\sum_{s^{t+1}|s^t} p_{t+1}(s^{t+1})}{p_t(s^t)} = \sum_{s^{t+1}} q_t(s^t, s_{t+1})$$

Realized return:

$$R_{t+1}^B(s^{t+1}) = \frac{1}{P_t^B(d, s^t)} = \frac{1}{\sum_{s_{t+1}} q_t(s^t, s_{t+1})} = R_{t+1}^B(s^t)$$

iii. Stock that pays as dividend the aggregate andowment per period (lucas tree):

$$P_t^S(d, s^t) = \frac{\sum_{\tau=t+1}^{\infty} \sum_{s^\tau|s^t} p_\tau(s^\tau) e_\tau(s^\tau)}{p_t(s^t)}$$

iv. Option to buy and sell a share of the Lucas tree at time T:

to buy: Call option $\quad P_t^{call} = \sum_{s^T|s^t} \frac{P_T(s^T)}{P_t(s^t)} \max\{P_T^S(d, s^T) - K, 0\}$

to sell: Put option $\quad P_t^{put} = \sum_{s^T|s^t} \frac{P_T(s^T)}{P_t(s^t)} \max\{K - P_T^S(d, s^T), 0\}$

15.1.1 Markov Processes (methodology)

In this context the analysis is implified by assuming the economy follows a discrete time, discrete state time homogenous markov chain: i.e. only the last state matters to define the probability of landing into the next state in the next period.

$$\pi(j|i) = Pr(s_{t+1} = j | s_t = i)$$

Transition matrix: $\quad \pi_t$ with π_{ij} : Probability of landing in state j given the preceding state is i.

Let $P_t = (p_t^1, \cdots, p_t^N)^T$ be the probability distribution over the N states, then the probability of being in state j tomorrow is,

$$p_{t+1}^j = \sum_i \pi_{ij} p_t^i$$

i.e. the sum of the conditional probabilities of going to state j from state i weighted by the probabilities of starting out in state i today.

More compactly,

$$P_{t+1} = \pi^T P_t$$

i.e. tomorrow's distribution is yesterday's multiplied by the transition matrix

homogenous chain: $\quad \pi_t = \Pi \quad \forall t$ (the transition matrix doesn't change in time)

Stationary distribution: $\quad \Pi \in \mathbb{R}_+^N$ such that,

$$\Pi = \pi' \Pi$$

Proposition: Associated with every markov transition matrix π there is at least one stationary distribution Π, it will be the eigenvector associated with the eigenvalue $\lambda = 1$ of π

Under these assumptions the probability of an event history given an initioal state at $t = 0$ is given by the product of one-step probabilities:

$$\pi_{t+1}(s^{t+1}) = \pi(s_{t+1}|s_t) \times \pi(s_t|s_{t-1}) \times \pi(s_{t-2}|s_{t-3}) \times \cdots \pi(s_1|s_0) \times \Pi(s_0)$$

Example: Suppose $N = 2$, let $p \in (0, 1)$ and the transition matrix be:

$$\pi = \begin{pmatrix} p & 1-p \\ 1-p & p \end{pmatrix}$$

the unique invariant distribution is $\Pi(s) = 0.5$:

$$\begin{aligned} \left(\overset{1}{\cancel{\lambda}} I - \pi\right) \mathbf{v} &= \left[\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} p & 1-p \\ 1-p & p \end{pmatrix}\right] \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \\ & \begin{pmatrix} 1-p & -1+p \\ -1+p & -p \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \\ & \begin{pmatrix} 1-p & -1+p \\ 0 & -1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \end{aligned}$$

in the last line the row1 is added to the row2 in the matrix. Then we get:

$$(1-p)v_1 - (1-p)v_2 = 0 \quad \Rightarrow \quad v_1 = v_2$$

Finally use that $v_1 + v_2 = 1$ to get $v_1 = v_2 = \frac{1}{2}$, i.e.,

$$\Pi(s) = \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}$$

Example: Let

$$\pi = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

then any distribution over the two states is an invariant distribution:

$$[I\lambda - \pi] \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

then the only restriction is that $v_1 + v_2 = 1$, i.e., any distribution works.

15.2 Stochastic NGM: The RBC model

- three goods: labor n_t, capital k_t, output y_t
- $\boxed{y_t = e^{z_t} F(k_t, n_t)}$
- z_t is an stochastic technology shock with mean zero and follows a N state markov chain:

$$Z = \{z_1, \cdots, z_N\}$$

$$\pi(z'|z) = Pr(z_{t+1} = z'|z_t = z)$$

- use $N = 2$
- capital dynamics: $k_{t+1} = (1-\delta)k_t + i_t$
- Preferences: $E_0 \sum_{t=0}^{\infty} \beta^t u(c_t)$
- Endowment: initial k_0 and time $n_t = 1$
- z_t is public info and z_0 is drawn from Π
- The state of the economy is given by (k, z) and not only k as before

SPP in recursive formulation

the stated assumptions and the observation that $n_t = 1$ will be optimal (no utility in leisure) leads to use the following Belman Equation:

$$v(k,z) = \max_{0 \le k' \le e^z F(k,1) + (1-\delta)k)} \left\{ U(\underbrace{e^z F(k,1) + (1-\delta)k - k'}_{c}) + \beta \sum_{z'} \pi(z'|z) v(k', z') \right\}$$

Now to allow for lifecycle behaviour we must include labor input fluctuations, hence agents will now value leisure so that the utility function becomes:

$$U(c_t, l_t) = U(c_t, 1 - n_t)$$

this is the Real Business Cycle (RBC) model, called it that way because it regards the fluctuations in economic activity to real shocks, in this case total factor productivity e^z. The new BE is therefore,

$$v(k,z) = \max_{\substack{0 \le k' \le e^z F(k,n) + (1-\delta)k) \\ 0 \le n_t \le 1}} \left\{ U(e^z F(k,n) + (1-\delta)k - k', 1 - n) + \beta \sum_{z'} \pi(z'|z) v(k', z') \right\}$$

FOC:

$[n]$: $$U_1 \cdot e^z F_n + U_2(-1) = 0 \quad \Rightarrow \quad \frac{U_2(c, 1-n)}{U_1(c, 1-n)} = e^z F_n(k,n) \qquad \text{[Intra]}$$

$[k']$: $$U_1 = \beta \sum_{z'} \pi(z'|z) v'(k', z')$$

where $v'(\cdot)$ is the derivative of the value function w.r.t. its first argument.

we use the envelope condition: $\frac{\partial v}{\partial k} = \frac{\partial U}{\partial k}$ and get:

$$v'(k,z) = (e^z F_k(k,n) + 1 - \delta)\, U_1(c, 1-n)$$

subs. in $[k']$:

$$U_1(c, 1-n) = \beta \sum_{z'} \pi(z'|z)\, (e^z F_k(k', n') + 1 - \delta)\, U_1(c', 1 - n') \qquad \text{[Inter-EE]}$$

Intuitions behind the equilibrium conditions:

Intra-temporal trade-off: Optinmal MRS between consumption and leisure is equal to marginal product of labor.

Inter-temporal trade-off: Euler equation, marginal utility of consumption today is equal to discounted marginal utility of consumption tomorrow $\underline{\text{adjusted by rate of return of capital}}$.

Recursive Competitive Equilibrium

value function $v : \mathbb{R}^3_+ \to \mathbb{R}_+$ and policy functions $c, n, g : \mathbb{R}^3_+ \to \mathbb{R}_+$ for the representative HH, a labor demand function for the representative firm $N : \mathbb{R}^2_+ \to \mathbb{R}_+$, pricing functions $w, r : \mathbb{R}^2_+ \to \mathbb{R}_+$ and an aggregate law of motion $H : \mathbb{R}^2_+ \to \mathbb{R}_+$ such that:

1. Given the functions w, r and H the value function solves the BE.

$$\max_{c,k',n\geq 0} \left\{ U(c,n) + \beta \sum_{z'\in Z} \pi(z'|z) v(k', z', K') \right\}$$

s.t.

$$c + k' = w(z,K)n + (1 + r(z,K) - \delta)k$$
$$K' = H(z,K)$$

with c, n, g as the associated policy functions.

2. The labor demand and pricing functions satisfy:

$$w(z,K) = e^z F_n(K, N(z,K))$$
$$w(z,K) = e^z F_k(K, N(z,K))$$

3. Consistency

$$H(z,K) = g(K,z,K)$$

4. MC: For all $K \in \mathbb{R}_+$

$$c(K,z,K) + g(K,z,K) = e^z F(K, N(z,K)) + (1-\delta)K$$
$$N(z,K) = n(K,z,K)$$