MS Macroeconomics – Tech Slides 2
Constrained Optimization Refresher and Monopolistic Competition

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Outline

I. Refresher of Constrained Optimization

II. Monopolistic Competition

Constrained Optimization (with equality constraints)

Commonly used in economics. The reason being that we care a lot about making the best (optimizing) out of limited resources (constraints)

Examples: Maximizations: how to allocate our time between studying and entertainment, how to spend our wage income. Minimizations: How to lower our commuting cost the most (given we have to go somewhere regularly).

Consider the problem:

$$\text{Extremize}_{x_1,\cdots,x_n} f(x_1,\cdots,x_n)$$

$$\text{subject to } g^j(x_1,\cdots,x_n) = b_j,\ \text{with } j = 1,\cdots,m;\ (m < n)$$

"Extremize" means that we want to either minimize or maximize the objective function $f(\cdot)$

Here: f is the objective function, g^j are the constraint functions, and b_j are the constraint constants.

Naive approach: Solve the problem without the constraints and then check if these are satisfied.

what we did in topic 2 (RBC)

Better approach: (often used in economics) replace the constraints in the objective function and solve the "unconstrained" version of the new problem. Convenient, but oftentimes not feasible.

Then, more generally, we resort to the ***Lagrangian***, or the method of ***Lagrange Multipliers***.

Setting the Lagrangian

The Lagrangian is given by a modification of the objective function that accounts for the constraints:

$$\mathcal{L} = f(x_1, \cdots, x_n) + \sum_{j=1}^{m} \lambda_j (b_j - g^j(x_1, \cdots, x_n))$$

Where the constants $\lambda_1, \cdots, \lambda_m$ are called *Lagrange Multipliers*.

How to solve: 0. Set $\mathcal{L}$

1. Take the partial derivatives of $\mathcal{L}$ with respect to $x_1, \cdots, x_n$ and $\lambda_1, \cdots, \lambda_m$ and equal to zero (FOC: First Order Conditions)
2. Use resulting system of equations to solve for the (optimal) values of $x_1, \cdots, x_n$ and $\lambda_1, \cdots, \lambda_m$

At the end we get the stationary points of the Lagrangian: $x^* = (x_1^*, x_2^* \cdots, x_n^*)$. Under certain conditions, this is the solution of the problem.

Note: This is the simplest case, hence the "certain conditions". The FOC yields necessary conditions that satisfy the solution, however, we should check Second order conditions to check for sufficient conditions. However, for the economic application of this lecture we just need to consider FOC.

First Order Conditions

This is how the FOC looks:

$$\frac{\partial \mathcal{L}}{\partial x_i} = \frac{\partial f(x_1, \cdots, x_n)}{\partial x_i} - \sum_{j=1}^{m} \frac{\partial g^j(x_1, \cdots, x_n)}{\partial x_i} = 0$$

and

$$\frac{\partial \mathcal{L}}{\partial \lambda_j} = b_j - g^j(x_1, \cdots, x_n) = 0$$

for all $i = 1, \cdots, n$ and $j = 1, \cdots, m$.

Note that the second buch of equations, once equalized to zero and arranged will be the constraints themselves ($b_j = g^j$). That's why in many econ courses and applications we don't explicitely take these derivatives, instead, we keep the (budget) constraints in the system of equations we solve at the end.

Example

$$\max_{x,y} U(x,y) = (x+1)(y+1) = xy + x + y + 1$$
$$\text{subject to } x + 2y \leq 30$$
$$x \geq 0, y \geq 0$$

Here the utility increases in x and y, then we can just assume the constraint binds (is satisfied with equality) and the quantities of both commodities are positive.

Then we can just solve: $\max_{x,y} U(x,y) \quad \text{s.t.} \quad x + 2y = 30$

Then the lagrangian is: $\mathcal{L} = (x+1)(y+1) + \lambda(30 - x - 2y)$

FOCs:

$$\frac{\partial \mathcal{L}}{\partial x} = y + 1 - \lambda = 0$$
$$\frac{\partial \mathcal{L}}{\partial y} = x + 1 - 2\lambda = 0$$
$$\frac{\partial \mathcal{L}}{\partial \lambda} = 30 - x - 2y = 0$$

from the first equation: $\lambda = y + 1$

Subst. λ in the second equation to get: $x - 2y - 1 = 0.$

Use this equation and the third one to get: $x^* = \frac{31}{2}, y^* = \frac{29}{4}$

Monopolistic Competition

blend of
- Perfect Competition → Firms are price takers
- vs.
- Monopolistic Power → each firm has Price Setting Power

The tool we just reviewed is used extensively in economics (a lot of econ is about picking allocation of recourses subject to scarcity constraints).

We covered it because we will use this tool repeatedly when reviewing the following market structure: **Monopolistic Competition**.

In a nutshell: we want to model a market where producers can act as monopolists (hence the "*monopolistic*") but are still affected by the decisions of other firms (hence the "*competition*")

Examples: many! actually, easier to find than for other market structures. Think of magazine markets, videogames, websites, fast food, etc.

Basically, any producer that owns a unique product (e.g. Mc Donalds) but still faces some competition by other firms that supply substitute products (e.g. Burger King) works as an example.

Set up

- Large number of households indexed and sorted on the unit interval $[0, 1]$
  (e.g., C_0, $C_{0.25}$ is the consumption of households "0" and "0.25")

- Each household **owns a firm** and produces a **differenti**ated variety of goods.

- Households want to consume a positive quantity of each type of good (love for variety).
  - Then the consumption basket is an aggregate of all available goods: $C_i = \underbrace{\left[\int_0^1 C_{ij}^{\frac{\sigma-1}{\sigma}} d_j\right]^{\frac{\sigma}{\sigma-1}}}_{\text{CES aggregator}}$

- Goods are **imperfect substitutes**

  Implications: Each firm (household producer) will act as monopolist, but the demand they face is also affected by other firms behavior (prices) (**monopolistic competition**)

- C_i is defined a CES aggregate (as above). The associated CPI is given by: $P_i = \underbrace{\left[\int_0^1 P_{ij}^{1-\sigma} dj\right]^{\frac{1}{1-\sigma}}}$

- The CES displays an elasticity of substitution between any two varieties equal to $\boxed{\sigma > 1}$. The higher σ is, the more substitutable the goods are. This elasticity is constant
  i.e., does not depend on the allocation, time or other variables.

  Elasticity of Substitution

Set up (cont.)

- Each firm produces its good using its own labor. Supplying labor (working) causes disutility
- holding (real) money provides utility.

  Then, the **utility** can be described by: $U(C_i, \frac{M_i}{P}, L_i) = \left(\frac{C_i}{\alpha}\right)^{\alpha} \left(\frac{M_i/P}{1-\alpha}\right)^{1-\alpha} - \frac{L_i^{\beta}}{\beta}$

  Where: C_i: consumption of househould i, M_i: (nominal) money holdings of household i, L_i: Labor supply of household i, P: General price level (CPI).

- Households resources consist of their sales revenues (P_iY_i) plus their initial money holdings ($\bar{M}_i$). They will use these resources for buying consumption goods and increasing their money holdings: (**Budget constraint**) $\int_0^1 P_j C_{ij} d_j + M_i = P_i Y_i + \bar{M}_i$

  Spending = Resources

- Technology: $Y_i = L_i$ (simple production function, production increases with the factor used).

Decisions of household i:

Then, the problem solved by the household i consists on determining the following quantities: C_{ij}

optimal demand of each good j (C_{ij}), the total consumption (C_i) and money holdings (M_i) C_i, M_i

Finally, the problem solved by the firms is to decide on their price level P_i

Firms decide: P_i

Solution method

We will divide the solution procedure in three steps:

1. Solve for the consumption demand of each good (variety) j: C_{ij}
2. Solve for the household allocation between consumption and money: C_i, M_i
3. Solve for the production and pricing decisions (here we assume homogenenous firms)

1. Solving for C_{ij}

This step is specific to the CES formulation, i.e., does not depend on the particular utility function choice.

Household i will maximize consumption subject to a spending level X_i (given).

Max C_i s.b. Resources to Spend on Consumption

$$\max_{C_{ij}} \underbrace{\left[\int_0^1 C_{ij}^{\frac{\sigma-1}{\sigma}} dj\right]^{\frac{\sigma}{\sigma-1}}}_{C_i} \text{ s.t. } \int_0^1 P_j C_{ij} dj = X_i$$

A simpler problem that yields the exact same solution is:

Max $C_i^{\frac{\sigma-1}{\sigma}}$ s.b. Resources to Spend on Consumption

$$\max_{C_{ij}} \underbrace{\int_0^1 C_{ij}^{\frac{\sigma-1}{\sigma}} dj}_{C_i^{\frac{\sigma-1}{\sigma}}} \text{ s.t. } \int_0^1 P_j C_{ij} dj = X_i$$

We set the lagrangian and get the first order conditions:

$$\mathcal{L} = \int_0^1 C_{ij}^{\frac{\sigma-1}{\sigma}} dj + \lambda \left(X_i - \int_0^1 P_j C_{ij} dj \right)$$

FOC:
wrt variety j

$$\boxed{[C_{ij}] : \quad \frac{\sigma-1}{\sigma} C_{ij}^{\frac{-1}{\sigma}} = \lambda P_j}$$

1. Solving for C_{ij} (cont.)

Notice that this holds for any $j \in [0,1]$. Then let's consider this condition for any two goods j and k:

$$\text{for j:} \quad \frac{\sigma-1}{\sigma} C_{ij}^{\frac{-1}{\sigma}} = \lambda P_j \qquad \text{for k:} \quad \frac{\sigma-1}{\sigma} C_{ik}^{\frac{-1}{\sigma}} = \lambda P_k$$

Divide these expressions:

$$\frac{C_{ik}}{C_{ij}} = \left(\frac{P_k}{P_j}\right)^{-\sigma} \tag{1}$$

Rewrite (1): $$\left(\frac{C_{ik}}{C_{ij}}\right)^{\frac{\sigma-1}{\sigma}} = \left(\frac{P_k}{P_j}\right)^{1-\sigma}$$

integrate over k:

$$\int_0^1 \left(\frac{C_{ik}}{C_{ij}}\right)^{\frac{\sigma-1}{\sigma}} dk = \int_0^1 \left(\frac{P_k}{P_j}\right)^{1-\sigma} d_k$$

$$\frac{1}{C_{ij}^{(\sigma-1)/\sigma}} \underbrace{\int_0^1 C_{ik}^{\frac{\sigma-1}{\sigma}} dk}_{C_i^{\frac{\sigma-1}{\sigma}}} = \frac{1}{P_j^{1-\sigma}} \underbrace{\int_0^1 P_k^{1-\sigma} dk}_{P^{1-\sigma}}$$

$$\boxed{\frac{C_i^{\frac{\sigma-1}{\sigma}}}{C_{ij}^{\frac{\sigma-1}{\sigma}}} = \frac{P^{1-\sigma}}{P_j^{1-\sigma}}}$$

when integrating over any variety "j" (or "k") we used that from definitions of C_i and P_i: $C_i^{(\sigma-1)/\sigma} = \int_0^1 C_{ij}^{(\sigma-1)/\sigma} dj$ and $P^{1-\sigma} = \int_0^1 P_j^{1-\sigma} dj$

1. Solving for C_{ij} (cont.)

- Rearrange and solve for the individual demand (variety ~~k~~ j):

$$\boxed{C_{ij} = \left(\frac{P_j}{P}\right)^{-\sigma} C_i} \tag{2}$$

Intuition: Demand for good j is proportional to total consumption, and inversely proportional to its relative price

Note: the inverse relationship is regulated by how substitutable is the good.

2. Allocation between C_i and M_i

From (2) it also follows (by multiplying by P_j and integrating over j) that $X_i = PC_i$. Use this to rewrite the Utility Maximization Problem:

$$\max_{C_i, M_i, L_i} \overbrace{\left(\frac{C_i}{\alpha}\right)^{\alpha}\left(\frac{M_i/P}{1-\alpha}\right)^{1-\alpha} - \frac{L_i^{\beta}}{\beta}}^{U(C_i, \frac{M_i}{P}, L_i)} \quad \text{s.t.} \quad \overbrace{C_i P}^{X_i} + M_i = P_i Y_i + \bar{M}_i$$

This utility function is a Cobb-Douglas aggregate (in C_i and M_i).

A standard result for a CD function is: Optimal Expenditure in A $= \frac{\text{CD coef. for A}}{\text{Sum of CD coefs}}$ Total Income

$A^{\alpha} B^{\beta}$, income I

$P_A \cdot A^* = \frac{\alpha}{\alpha+\beta} \cdot I \rightarrow A^* = \frac{\alpha}{\alpha+\beta} \frac{I}{P_A}$

$B^* = \frac{\beta}{\alpha+\beta} \cdot \frac{I}{P}$

Then given $Y_i, \bar{M}_i$, the CD solutions for our quantities of interest are:

algebra steps here

$$PC_i = \alpha(P_i Y_i + \bar{M}_i) \qquad M_i = (1-\alpha)(P_i Y_i + \bar{M}_i)$$

2. Allocation between C_i and M_i (cont.)

From these expressions we have: $\boxed{C_i = \frac{\alpha}{1-\alpha}\frac{M_i}{P}}$, plug it in (2):

(2): $C_{ij} = \left(\frac{P_j}{P}\right)^{-\sigma} \cdot C_i$

$$C_{ij} = \frac{\alpha}{1-\alpha}\frac{M_i}{P}\left(\frac{P_j}{P}\right)^{-\sigma} \tag{3}$$

This is the optimal decision rule for consumption of good j v.s. increasing money holdings (saving).

Consumption of variety j decreases in relative price and in money holdings.

3. Production and Pricing decisions

Replace C_i, M_i and $Y_i = L_i$ in the utility function:

$$U = \left(\frac{\cancel{\alpha}}{P} \frac{(P_iY_i + \bar{M}_i)}{\cancel{\alpha}} \right)^{\alpha} \left(\frac{1}{P} \frac{1}{\cancel{1-\alpha}} \cancel{(1-\alpha)} (P_iY_i + \bar{M}_i) \right)^{1-\alpha} - \frac{Y_i^{\beta}}{\beta} \quad \Rightarrow \quad \boxed{U = \frac{P_iY_i}{P} - \frac{Y_i^{\beta}}{\beta} + \frac{\bar{M}_i}{P}}$$

Total demand for each good i is the sum over the k households demands:

$$Y_i = \int_0^1 C_{ki} dk = \int_0^1 \frac{\alpha}{1-\alpha} \frac{M_k}{P} \left(\frac{P_i}{P} \right)^{-\sigma} dk = \frac{\alpha}{1-\alpha} \frac{1}{P} \left(\frac{P_i}{P} \right)^{-\sigma} \underbrace{\int_0^1 M_k dk}_{M} \quad \Rightarrow \quad \boxed{\mathbf{Y_i} = \frac{\alpha}{\mathbf{1}-\alpha} \frac{\mathbf{M}}{\mathbf{P}} \left(\frac{\mathbf{P_i}}{\mathbf{P}} \right)^{-\sigma}}$$

replace from (3)

Thus we get the typical downward sloping AD curve $(Q_i = D_i(\frac{P_i}{P}, \frac{M}{P}))$

3. Production and Pricing decisions (cont.)

Maximizing the new transformed $U(\cdot)$ gives (from the FOC):

steps here

$$\frac{P_i}{P} = \boxed{\frac{\sigma}{\sigma - 1}} Y_i^{\beta - 1}$$

Price = markup x marginal cost

Mark-up > 1

Replace Y_i:

$$\frac{P_i}{P} = \left[\frac{\sigma}{\sigma - 1} \left(\frac{\alpha}{1 - \alpha} \right)^{\beta - 1} \left(\frac{M}{P} \right)^{\beta - 1} \right]^{\frac{1}{1 + \sigma(\beta - 1)}}$$

$\therefore$ when ΔM the effect on Y and $\frac{P_i}{P}$ depends on β, σ.
If $\beta = 1$ the relative price doesn't change and ΔM is accomodated by ΔY.

As $\beta > 1$, ΔM leads to $\Delta \frac{P_i}{P}$ together with quantity adjustment.

General Equilibrium

Now we consider the implications for the general output and level of prices.

For that, a key assumption we use is that **households and firms are symmetric**, i.e., they all take analogous decisions: $Y_i = Y_j = Y, P_i = P_j = P$.

With symmetric HH relative price is 1:

$$\boxed{1 = \frac{\sigma}{\sigma - 1} Y^{\beta - 1} \quad P = \frac{\alpha}{1-\alpha} \frac{M}{Y} \propto M}$$

$\frac{P_i}{P} = 1$

Output is a constant markup and **money is neutral**, i.e., $\Delta M \Rightarrow \Delta P$ proportionally and the effect on Y is null. prices accomodates to undo the effect of money

Does this mean Money is always neutral?

Not always, e.g., with **menu costs** the models implies **monetary non neutrality**.

Intuition: ΔM small then prices won't change (not worth assuming the cost) $\bar{P} \Rightarrow$ non neutrality.

More technically, deviations from optimal price have only second order effects on profits (then firms don't adjust their price) whereas first and second on social welfare (then output increases).

Derivation optimal consumption-money allocation (step 2)

$$\mathcal{L} = \left(\frac{C_i}{\alpha}\right)^{\alpha} \left(\frac{M_i/P}{1-\alpha}\right)^{1-\alpha} - \frac{L_i^{\beta}}{\beta} + \lambda(P_i Y_i + \bar{M}_i - C_i P - M_i)$$

FOC:

$$[C_i]: \qquad \alpha \left(\frac{C_i}{\alpha}\right)^{\alpha} \left(\frac{M_i/P}{1-\alpha}\right)^{1-\alpha} C_i^{-1} = \lambda P$$

$$[M_i]: \qquad \left(\frac{C_i}{\alpha}\right)^{\alpha} \cancel{(1-\alpha)} \left(\frac{M_i/P}{1-\alpha}\right)^{-\alpha} \frac{1}{\cancel{(1-\alpha)}} \frac{1}{P} = \lambda$$

$$\frac{1}{P} \left(\frac{C_i}{\alpha}\right)^{\alpha} \left(\frac{M_i/P}{1-\alpha}\right)^{-\alpha} \left(\frac{M_i/P}{1-\alpha}\right) \left(\frac{M_i/P}{1-\alpha}\right)^{-1} = \lambda$$

$$(1-\alpha) \left(\frac{C_i}{\alpha}\right)^{\alpha} \left(\frac{M_i/P}{1-\alpha}\right)^{1-\alpha} \left(\frac{M_i/P}{1-\alpha}\right)^{-1} = \lambda P$$

Divide $[C_i]$ by $[M_i]$:

$$\frac{\alpha \cancel{\left(\frac{C_i}{\alpha}\right)^{\alpha} \left(\frac{M_i/P}{1-\alpha}\right)^{1-\alpha}} C_i^{-1}}{(1-\alpha) \cancel{\left(\frac{C_i}{\alpha}\right)^{\alpha} \left(\frac{M_i/P}{1-\alpha}\right)^{1-\alpha}} (M_i/P)^{-1}} = \frac{\lambda \cancelto{1}{P}}{\cancel{\lambda} P}$$

Then: $\frac{\alpha}{1-\alpha} \frac{M_i/P}{C_i} = 1$, or $\alpha \frac{M_i}{P} = (1-\alpha) C_i$

From this expression substitute M_i in the budget constraint:

$$\begin{aligned} PC_i + M_i &= P_i Y_i + \bar{M}_i \\ PC_i + \tfrac{(1-\alpha)}{\alpha} PC_i &= P_i Y_i + \bar{M}_i \\ \tfrac{1}{\alpha} PC_i &= P_i Y_i + \bar{M}_i \\ PC_i &= \alpha(P_i Y_i + \bar{M}_i) \end{aligned}$$

Similarly, plug PC_i in the budget constraint:

$$\begin{aligned} \alpha(P_i Y_i + \bar{M}_i) + M_i &= P_i Y_i + \bar{M}_i \\ M_i &= (1-\alpha)(P_i Y_i + \bar{M}_i) \end{aligned}$$

back

Derivation of optimal pricing rule (step 3)

Replacing Y_i in the optimization problem:

Pricing decision

(s.t. optimal total demand for good Y_i)

$$\max_{P_i} \frac{P_i}{P}\underbrace{\frac{\alpha}{1-\alpha}\frac{M}{P}\left(\frac{P_i}{P}\right)^{-\sigma}}_{Y_i} - \left(\underbrace{\frac{\alpha}{1-\alpha}\frac{M}{P}\left(\frac{P_i}{P}\right)^{-\sigma}}_{Y_i}\right)^{\beta}\frac{1}{\beta} + \frac{\bar{M}_i}{P}$$

Rearrange:

$$\max_{P_i} \frac{\alpha}{1-\alpha}\frac{M}{P}\left(\frac{P_i}{P}\right)^{1-\sigma} - \left(\frac{\alpha}{1-\alpha}\frac{M}{P}\left(\frac{P_i}{P}\right)^{-\sigma}\right)^{\beta}\frac{1}{\beta} + \frac{\bar{M}_i}{P}$$

FOC:

$$[P_i]: \qquad \frac{\alpha}{1-\alpha}(1-\sigma)\left(\frac{P_i}{P}\right)^{-\sigma}\frac{1}{P}\frac{M}{P} - \beta\left(\frac{\alpha}{1-\alpha}\frac{M}{P}\left(\frac{P_i}{P}\right)^{-\sigma}\right)^{\beta-1}\frac{1}{\beta}\frac{\alpha}{1-\alpha}\frac{M}{P}(-\sigma)\left(\frac{P_i}{P}\right)^{-\sigma-1}\frac{1}{P} = 0$$

Simplify and put back Y_i: $\qquad (1-\sigma)Y_i\cancel{\tfrac{1}{P}} + \sigma Y_i^{\beta-1}Y_i\left(\frac{P_i}{P}\right)^{-1}\cancel{\tfrac{1}{P}} = 0$ Then:

$$\frac{P_i}{P} = \frac{\sigma}{1-\sigma}Y_i^{\beta-1}$$

back