# Problem set # 4

Answer Key

1. **(Monopolistic Competition with taste shocks)** (this follows Dixit and Stiglitz, 1997) Suppose that the consumption index in equation $U_i = C_i - \frac{1}{\gamma} L_i^{\gamma}$ is $C_i = \left[\int_{j=0}^{1} Z_j^{\frac{1}{\eta}} C_{ij}^{(\eta-1)/\eta} dj\right]^{\frac{\eta}{\eta-1}}$, where $C_{ij}$ is the individual's consumption of good $j$ and $Z_j$ is the taste shock for good $j$. Suppose the individual has amount $Y_i$ to spend on goods. thus the budget constraint is $\int_{j=0}^{1} P_j C_{ij} dj = Y_i$.

   (a) Find the first-order condition for $C_{ij}$ for the problem of maximizing $C_i$ subject to the budget constraint. Solve for $C_{ij}$ in terms of $Z_j$, $P_j$, and the Lagrange multiplier on the budget constraint.

(Ans)

$$\mathcal{L} = \underbrace{\left[\int_{j=0}^{1} Z_j^{\frac{1}{\eta}} C_{ij}^{\frac{\eta-1}{\eta}} d_j\right]^{\frac{\eta}{\eta-1}}}_{C_i} + \lambda\left(Y_i - \int_0^1 P_j C_{ij} dj\right)$$

F.O.C.:

$$[C_{ij}]: \quad \frac{\eta}{\eta-1}\left[C_i^{\frac{\eta-1}{\eta}}\right]^{\frac{\eta}{\eta-1}-1} \frac{\eta-1}{\eta} Z_j^{\frac{1}{\eta}} C_{ij}^{-\frac{1}{\eta}} - \lambda P_j = 0 \tag{1}$$

Now we solve for $C_{ij}$:

$$\left[C_i^{\frac{\eta-1}{\eta}}\right]^{\frac{1}{\eta-1}} C_{ij}^{-\frac{1}{\eta}} = \lambda Z_j^{-\frac{1}{\eta}} P_j$$

$$C_{ij}^{-\frac{1}{\eta}} = C_i^{-\frac{1}{\eta}} \lambda Z_j^{-\frac{1}{\eta}} P_j$$

$$C_{ij} = C_i \lambda^{-\eta} Z_j P_j^{-\eta}$$

This is the most simplified expression we can get to. Notice we haven't gotten rid of the aggregate consumption of the household ($C_i$) or of the lagrange multiplier.

In this case $C_i$ is showing up because we are maximizing total consumption, whereas in the tech session and lecture handouts we (simplified) changed the problem slightly and maximized $C_i^{\frac{\eta-1}{\eta}}$. If you used the simplified version the resulting equation is what the problem is asking for (a), but I will provide full credit if you arrived to this equation too.

In order to simplify further and get rid of the lagrange multiplier we will consider this F.O.C. for another good variety $k$.

   (b) Use the budget constraint to find $C_{ij}$ in terms of $Z_j$, $P_j$, $Y_i$, and the $Z$'s and $P$'s.

(Ans) Consider the F.O.C. or the simplified equation below for $C_{ik}$:

$$C_{ik} = C_i \lambda^{-\eta} Z_k P_k^{-\eta}$$

Divide the expression for $C_{ij}$ by that of $C_{ik}$ and rearrange:

$$C_{ij} = \frac{Z_j}{Z_k} \left( \frac{P_j}{P_k} \right)^{-\eta} C_{ik}$$

Now we replace from this expression $C_{ij}$ in the budget constraint:

$$Y_i = \int_0^1 P_j C_{ij} dj$$
$$Y_i = \int_0^1 P_j \frac{Z_j}{Z_k} \left( \frac{P_j}{P_k} \right)^{-\eta} C_{ik} dj$$
$$Y_i = \frac{C_{ik}}{Z_k P_k^{-\eta}} \int_0^1 Z_j P_j^{1-\eta} dj$$

Then,

$$C_{ik} = \frac{Z_k P_k^{-\eta} Y_i}{\int_0^1 Z_j P_j^{1-\eta} dj}$$

This equation holds for any good variety. Then we can re-express it in terms of our quantity (good variaty) of interest $C_{ij}$:

$$C_{ij} = \frac{Z_j P_j^{-\eta} Y_i}{\int_0^1 Z_k P_k^{1-\eta} dk} = \frac{Z_j P_j^{-\eta} Y_i}{\int_0^1 Z_j P_j^{1-\eta} dj}$$

Notice this is valid since $\int_0^1 Z_j P_j^{1-\eta} dj$ and $\int_0^1 Z_k P_k^{1-\eta} dk$ are identical quantities (just an aggregate through all possible variaties).

(c) Substitute your result in part $(b)$ into the expression for $C_i$ and show that $C_i = Y_i / P$, where $P = [\int_{j=0}^1 Z_j P_j^{1-\eta} dj]^{\frac{1}{1-\eta}}$.

(Ans) We set the total consumption expression and replace the result for $C_{ij}$ from (b). Then we rearrange:

$$C_i = \left[ \int_0^1 Z_j^{\frac{1}{\eta}} C_{ij}^{\frac{\eta-1}{\eta}} dj \right]^{\frac{\eta}{\eta-1}}$$
$$C_i^{\frac{\eta-1}{\eta}} = \int_0^1 Z_j^{\frac{1}{\eta}} \left[ \frac{Z_j P_j^{-\eta} Y_i}{\int_0^1 Z_j P_j^{1-\eta} dj} \right]^{\frac{\eta-1}{\eta}} dj$$
$$= \int_0^1 Z_j^{\frac{1}{\eta}} Z_j^{\frac{\eta-1}{\eta}} P_j^{1-\eta} Y_i^{\frac{\eta-1}{\eta}} \frac{1}{\left[ \int_0^1 Z_j P_j^{1-\eta} dj \right]^{\frac{\eta-1}{\eta}}} dj$$
$$= Y_i^{\frac{\eta-1}{\eta}} \left[ \int_0^1 Z_j P_j^{1-\eta} dj \right]^{\frac{-(\eta-1)}{\eta}} \int_0^1 Z_j P_j^{1-\eta} dj$$

$$= Y_i^{\frac{\eta-1}{\eta}} \left[\int_0^1 Z_j P_j^{1-\eta} dj\right]^{\frac{1}{\eta}}$$

In the fourth line, notice we can factor $\left[\int_0^1 Z_j P_j^{1-\eta} dj\right]^{\frac{-(\eta-1)}{\eta}}$ because (after integrated and evaluated between 0 and 1) it does not really depend on $j$.

Now we raise both sides to the power of $\frac{\eta}{\eta-1}$ and get an expression for consumption:

$$C_i = Y_i \left[\int_0^1 Z_j P_j^{1-\eta} dj\right]^{\frac{1}{\eta-1}} = Y_i P^{-1}$$

Where $P = \left[\int_0^1 Z_j P_j^{1-\eta}\right]^{\frac{1}{1-\eta}}$

(d) Use the results in part $(b)$ and part $(c)$ to show that $C_{ij} = Z_j (P_j/P)^{-\eta} (Y_i/P)$.

(Ans) From (b) we know:

$$C_{ij} = \frac{Z_j P_j^{-\eta} Y_i}{\int_0^1 Z_j P_j^{1-\eta} dj}$$

Also, $P = \left[\int_0^1 Z_j P_j^{1-\eta}\right]^{\frac{1}{1-\eta}}$, meaning the denominator in the ratio above is $P^{1-\eta}$.

Then:

$$C_{ij} = \frac{Z_j P_j^{-\eta} Y_i}{P^{1-\eta}} = Z_j \left(\frac{P_j}{P}\right)^{-\eta} \frac{Y_i}{P}$$

2. **(Calvo Handout)** Please go over the Calvo handout (on course website).

(a) Derive equation (x), equation above $(*)$, equation $(*)$, equation $(+)$ and the Calvo Phillips curve. Be sure to show your work (most relevant steps).

(Ans) The optimal price for any given firm is:

$$p_t^* = p_t + \phi y_t,$$

where $\phi y_t$ denotes the degree of real rididity (strategic complementarity in the price setting), and $p_t$ the aggregate price level. Notice that we are dropping the firm index as they are symmetric.

The profic loss from deviations of the optimal price for a firm that sets a price $x$ is:

$$\pi(p_t^*) - \pi(x_t) = \underbrace{\pi'(p_t^*)}_{0}(p_t^* - x_t) + \pi''(p_t^*)(p_t^* - x_t)^2 = \frac{\kappa}{2}(p_t^* - x_t)^2$$

We will take this loss function and set the expected total profit loss throughout the lifetime of the firm (that lasts infinite periods):

$$\min_{x_t} \sum_{j=0}^{\infty}(1-\theta)\theta^j\beta^j\frac{\kappa}{2}(\mathbb{E}_t p^*_{t+j} - x_t)^2$$

FOC:

$$\kappa(1-\theta)\sum_{j=0}^{\infty}\theta^j\beta^j(\mathbb{E}_t p^*_{t+j} - x_t) = 0$$

$$\sum_{j=0}^{\infty}\theta^j\beta^j\mathbb{E}_t p^*_{t+j} - \frac{1}{1-\theta\beta}x_t = 0$$

Where we used that $\sum_{j=0}^{\infty}(\theta\beta)^j = \frac{1}{1-\theta\beta}$ given that $0 < \theta\beta < 1$.

From this F.O.C. we get the expression for $x_t$:

$$x_t = (1-\theta\beta)\sum_{j=0}^{\infty}\theta^j\beta^j\mathbb{E}_t p^*_{t+j} \tag{x}$$

Now, forward the equation for $x_t$ to $t+1$, fix the sum indexes so it starts at $j=1$, multiply times $\theta\beta$, take expectations ($\mathbb{E}_t(\cdot)$), and rearrange:

$$x_{t+1} = (1-\theta\beta)\sum_{j=0}^{\infty}\theta^j\beta^j\mathbb{E}_{t+1} p^*_{t+1+j}$$

$$x_{t+1} = (1-\theta\beta)\sum_{j=1}^{\infty}\theta^{j-1}\beta^{j-1}\mathbb{E}_{t+1} p^*_{t+j}$$

$$\theta\beta x_{t+1} = (1-\theta\beta)\sum_{j=1}^{\infty}\theta^j\beta^j\mathbb{E}_{t+1} p^*_{t+j}$$

$$\theta\beta\mathbb{E}_t x_{t+1} = (1-\theta\beta)\sum_{j=1}^{\infty}\theta^j\beta^j\mathbb{E}_t p^*_{t+j},$$

where in the last line we applied the expectation operator conditional to $t$ ($\mathbb{E}_t(\cdot)$) on each side and used the law of iterated expectations in the last term: $\mathbb{E}_t\mathbb{E}_{t+1}p^*_{t+1} = \mathbb{E}_t p^*_{t+1}$.

Now we re-arrange the equation for $x_t$ to write it in a difference equation form:

$$\begin{aligned} x_t &= (1-\theta\beta)\theta^0\beta^0\mathbb{E}_t p^*_t + (1-\theta\beta)\sum_{j=1}^{\infty}\theta^j\beta^j\mathbb{E}_t p^*_{t+j} \\ &= (1-\beta\theta)p^*_t + \beta\theta\mathbb{E}_t x_{t+1} \end{aligned}$$

replace the definition of $p^*_t = p_t + \phi y_t$::

$$x_t = \theta\beta\mathbb{E}_t x_{t+1} + (1-\theta\beta)p_t + (1-\theta\beta)\phi y_t$$

rearrange to get: $x_t - p_t - \theta\beta\mathbb{E}_t x_{t+1} = -\theta\beta p_t + (1-\theta\beta)\phi y_t$. Then:

$$\begin{aligned} x_t - p_t - \theta\beta\mathbb{E}_t x_{t+1} + \theta\beta\mathbb{E}_t p_{t+1} &= \theta\beta\mathbb{E}_t p_{t+1} - \theta\beta p_t + (1-\theta\beta)\phi y_t \\ z_t - \theta\beta\mathbb{E}_t z_{t+1} &= \theta\beta\pi_{t+1} + (1-\theta\beta)\phi y_t \qquad (*) \end{aligned}$$

where $z_t = x_t - p_t$.

aggregrate prices are also given by:

$$p_t = (1-\theta)\sum_{j=0}^{\infty}\theta^j x_{t-j} = (1-\theta)x_t + \theta(1-\theta)\sum_{j=0}^{\infty}\theta^j x_{t-1-j}$$

i.e. $p_t = (1-\theta)x_t + \theta p_{t-1}$

Now we can get an expression for the inflation:

$$\begin{aligned} (1-\theta)p_t + \theta p_t &= (1-\theta)x_t + \theta p_{t-1} \\ \theta(p_t - p_{t-1}) &= (1-\theta)(x_t - p_t) \\ \theta\pi_t &= (1-\theta)z_t \qquad (+) \end{aligned}$$

forward, take $\mathbb{E}_t[\cdot]$ and multiply by $-\theta\beta$, getting $-\beta\theta^2\mathbb{E}_t\pi_{t+1} = -\beta\theta(1-\theta)\mathbb{E}_t z_{t+1}$, add that to the inflation eq:

$$(1-\theta)(z_t - \beta\theta\mathbb{E}_t z_{t+1}) = \theta(\pi_t - \beta\theta\mathbb{E}_t\pi_{t+1})$$

subs. $(*)$ in the LHS of this equation and rearrange:

$$\begin{aligned} (1-\theta)(\theta\beta\pi_{t+1} + (1-\theta\beta)\phi y_t) &= \theta(\pi_t - \beta\theta\mathbb{E}_t\pi_{t+1}) \\ \theta\beta\mathbb{E}_t\pi_{t+1} + (1-\theta)(1-\theta\beta)\phi y_t &= \theta\pi_t \\ \Rightarrow \qquad \pi_t &= \beta\mathbb{E}_t\pi_{t+1} + \frac{(1-\theta)(1-\theta\beta)}{\theta}\phi y_t \end{aligned}$$

The last like corresponds to the Phillips Curve:

$$\pi_t = \beta\mathbb{E}_t\pi_{t+1} + \lambda y_t$$

with $\lambda = \frac{(1-\theta)(1-\theta\beta)}{\theta}\phi$

(b) Provide intuitions for equation (x) and the Calvo Phillips curve. Interpret the coefficients $\beta$ and $\lambda$ in the Calvo Phillips Curve.

(Ans) Interpretation

(x): the optimal price at t ($x_t$ in (x)) is equal to the average price for the firms across all future periods, weigthed by the probability that the price remains effective during those periods. Remember, the probability of the price not changing for $n$ periods is $\theta^n$, also, this price is denoted in present value, hence the $\beta^j$.

Phillips Curve:

$\beta$: The inflation will depend on the present value of the expected inflation next period. If the inflation is expected to rise, the current inflation, after some firms reset prices will go up, but a bit less than proportionally given the generalized surge in prices haven been realized by time $t$.

$\lambda$: This coefficient represents the output-inflation trade-off, the larger, the lower the effect of the inflation in the ouput (remember the associated slope of the AS curve would be $\frac{1}{\lambda}$). This coefficient is larger for a higher extent of the real rigidity ($\lambda$ grows in $\phi$), and for higher extent of the nominal rigidity (i.e. $\lambda$ increases if $\theta$ grows and more firms are unable to adjust prices).

3. Explain whether the following statement is *true, false or uncertain*:

   (a) **(Menu Costs)** If menu costs are the reason for monetary non-neutrality, then according to these models, a recession caused by a monetary contraction would not be very costly.

(Ans) This question can be thought in terms of the Blanchard-Kiyotaki model.

This statement is not necessarily true (it could be said it's false or uncertain), if the potential profit gain from adjusting prices after the contraction is large enough, the firms will pay the menu costs (even if these are substantial) as, for them, the potential profit gain still outweighs bearing the cost. The society is hurt by the extent of that cost of a result.

If instead, firms don't change prices and produce with the old price, they will reduce the production, and the society will still incurr in a deadweight loss (triangle ABC in the plot).

5. **(Setting Dynare and Solving your first DSGE model.)** The objective of this question is to make sure you set up Dynare in your computer and start getting familiar with its basic use.

   (c) Report the plots the file generates (this will be the Impulse Response Functions of the endogenous variables after a shock in the model).

(Ans) The resulting plot is:

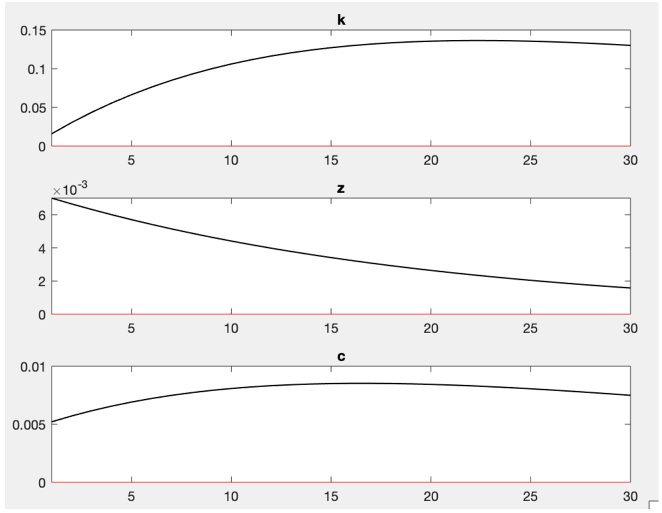


**Figure 1:** Impulse response of variables of the model to a 1 std. dev. shock in productivity