Econ 5322 - MS Macroeconomics — Spring 2024

Granados

Problem set # 0

Answer Key

1. Find $\frac{\partial f}{\partial x}$ for $f(x) = \frac{x^{1-\sigma}}{1-\sigma}$ where $\sigma > 0$ is a constant.

Ans: $\frac{\partial f}{\partial x} = \frac{1}{1-\sigma}(1-\sigma)x^{1-\sigma-1} = x^{-\sigma}$

2. Find $\frac{\partial f}{\partial x}$ when $f(x,y) = x^3 y \ln x$, where $\ln(\cdot)$ is the natural logarithm.

Ans: $\frac{\partial f(x,y)}{\partial x} = 3x^2 y \ln x + x^3 y \frac{1}{x} = yx^2(3\ln x + \frac{1}{x}) = yx^2(3\ln x + 1)$

3. If $f(x,\beta) = (\frac{x}{\beta^2})e^{-x/\beta}$, find $\ln f(x,\beta)$. Use this to find $\frac{\partial}{\partial \beta}\ln f(x,\beta)$

Ans: $\ln f(x,\beta) = \ln(\frac{x}{\beta^2}) + \ln e^{-x/\beta} = \ln x - 2\ln\beta - \frac{x}{\beta}$

Then: $\frac{\partial \ln f(x,\beta)}{\partial x} = \frac{1}{x} - \frac{1}{\beta}$ and $\frac{\partial f(x,\beta)}{\partial \beta} = -2\frac{1}{\beta} + \frac{x}{\beta^2}$

4. For $0 < \beta < 1$, show that $\sum_{t=0}^{\infty} \beta^t = \frac{1}{1-\beta}$

Ans:

Let s be the left hand side of the target expression:

$$s = \sum_{t=0}^{\infty} \beta^t = \beta^0 + \beta^1 + \beta^2 + \beta^3 \cdots$$

multiply both sides by $(1-\beta)$:

$$(1-\beta)s = \beta^0 - \beta^1 + \beta^1 - \beta^2 + \beta^2 - \beta^3 + \beta^3 + \cdots$$

$$(1-\beta)s = \cancel{\beta^0}^{\,1} - \cancel{\beta^1} + \cancel{\beta^1} - \cancel{\beta^2} + \cancel{\beta^2} - \cancel{\beta^3} + \cancel{\beta^3} + \cdots$$

$$(1-\beta)s = 1$$

Then: $s = \frac{1}{1-\beta} = \sum_{t=0}^{\infty} \beta^t$

5. **[Basic constrained optimization - application from microeconomics]**

Consider this optimization problem:[1]

$$\max_{x,y} U(x,y) = (x+1)(y+1) = xy + x + y + 1$$

$$\text{subject to } x + 2y = 30$$

Find the optimal levels of x, y that solve the optimization problem above.

For this, set the associated lagrangian, find the First Order Conditions and solve for x and y.

Ans:

The lagrangian associated to this problem is: $\mathcal{L} = (x+1)(y+1) + \lambda(30 - x - 2y)$.

The first order conditions obtained from equating the partial derivatives of $\mathcal{L}$ with respect to x, y, and the multiplier associated to the constraint (λ) and equating them to zero are:

$$\frac{\partial \mathcal{L}}{\partial x} = y + 1 - \lambda = 0$$
$$\frac{\partial \mathcal{L}}{\partial y} = x + 1 - 2\lambda = 0$$
$$\frac{\partial \mathcal{L}}{\partial \lambda} = 30 - x - 2y = 0$$

from the first equation: $\lambda = y + 1$

Subst. λ in the second equation to get $x - 2y - 1 = 0$.

Use this equation and the third one to get: $x^* = \frac{31}{2}$, $y^* = \frac{29}{4}$

6. **[Dynamic optimization first order condition]**

Consider the function $L = \sum_{s=0}^{\infty} \beta^{t+s} \left[\sqrt{x_{t+s}} + 35 + \lambda_{t+s}(100 - \frac{1}{2}x_{t+s} + \frac{4}{5}x_{t+s-1})\right]$

Find the derivative of this function with respect to x_t.

Ans:

L expressed as above is just a compact way to write the following equation:

$$\begin{aligned} L = & \beta^t[\sqrt{x_t} + 35 + \lambda_t(100 - \frac{1}{2}x_t + \frac{4}{5}x_{t-1})] \\ & + \beta^{t+1}[\sqrt{x_{t+1}} + 35 + \lambda_{t+1}(100 - \frac{1}{2}x_{t+1} + \frac{4}{5}x_t)] \\ & + \beta^{t+2}[\sqrt{x_{t+2}} + 35 + \lambda_{t+2}(100 - \frac{1}{2}x_{t+2} + \frac{4}{5}x_{t+1})] \\ & \vdots \end{aligned}$$

[1] The actual problem has as budget constraint $x + 2y \leq 30$, and non-negativity constraints for x and y. Based on economic theory we know we can solve the simpler problem above where the first constraint binds and the quantities consumed of each commodity is larger than zero.

Now that we know all the terms in the equation we just have to take $\frac{\partial L}{\partial x_t}$ as usual:

$$\frac{\partial L}{\partial x_t} = \beta^t \left(\frac{1}{2\sqrt{x_t}} - \frac{\lambda_t}{2} \right) + \beta^{t+1} \lambda_{t+1} \frac{4}{5}$$