Closed Economy NK DSGE

Case with linear production in labor

This handout contains the algebra for the standard closed economy Neo-Keynesian DSGE model. The model included here features nominal prices rigidigies, monopolistic competition in the intermediate goods sector, and intermediate firms that use labor as input (main derivations are in black and extra clarifications in gray).

Main references: Gali (2015, Ch. 5), Walsh (2007, Ch. 8), and Woodford (2003, Ch. 3), the structure of these notes also borrow from derivations made by Mutschler (U. Tubingen, Dynare development team).

Initial Definitions

It is useful to define the real bonds b_t and price of bonds Q_t as:

$$b_t = \frac{B_t}{P_t}$$
$$Q_t = \frac{1}{R_t}$$

Where B_t are the nominal bonds, P_t the level of prices, or price of the final consumption good, and R_t are the nominal interest rate of bonds.

Also, the real interest rate r_t and the nominal one are related through a Fisher equation:

$$r_t = \mathbb{E}_t \frac{R_t}{\Pi_{t+1}} \tag{1}$$

Where $\Pi_{t+1} = \frac{P_{t+1}}{P_t}$ is the gross inflation at $t+1$.

Households

The representative household will maximize the present value of its lifetime utility subject to a budget constraint by solving:

$$\max_{B_t, c_t,} \mathbb{E}_t \sum_{s=0}^{\infty} \beta^s u(c_{t+s}, n^s_{t+s}, z_{t+s})$$
$$s.t., \int_0^1 P_t(h) c_t(h) dh + Q_t B_t \leq B_{t-1} + P_t \int_0^1 div_t(f) df + W_t n^s_t \qquad \text{(-nominal- budget constraint)}$$

The budget constraint includes on the spending side the total spending in a variety of goods indexed by h (each one with a price $P_t(h)$ and a quantity consumed $c_t(h)$), and nominal bonds purchased at a price Q_t and that will pay a quantity B_t the next period. On the income side, the budget includes bonds purchased the period before, the nominal dividends or profits from intermediate firms owned (total real profits times the nominal price level) and the nominal labor income given by the nominal wage W_t times the labor supply n^s_t.

We will consider CRRA preferences given by:

$$u(c_t, n^s_t, z_t) = z_t \left(\frac{c_t^{1-\sigma}}{1-\sigma} - \frac{(n^s_t)^{1+\psi}}{1+\psi} \right)$$

where σ is the risk aversion parameter (or inverse Intertemporal Elasticity of Substitution), ψ is the Frisch labor desutility parameter, and z_t is a preferences shock (demand shock).

Also, we can simplify the nominal budget constraint and set it in real terms:

First, we replace the aggregate consumption spending:

$$P_t c_t = \int_0^1 P_t(h) c_t(h) dh$$

→ (detour) We can prove this result:

Consider the following assumptions,

(A1) the optimal consumption demand for variety h is: $c_t(h) = \left(\frac{P_t(h)}{P_t}\right)^{-\epsilon} c_t$

(A2) the price level of the associated aggregate consumption basket is $P_t = \left(\int_0^1 P_t(h)^{1-\epsilon} dh\right)^{\frac{1}{1-\epsilon}}$

Then replace the optimal demand in the aggregate consumption spending:

$$\begin{aligned}\int_0^1 P_t(h)c_t(h)dh &= \int_0^1 P_t(h)\left(\frac{P_t(h)}{P_t}\right)^{-\epsilon} c_t dh \\ &= \frac{c_t}{P_t^{-\epsilon}}\int_0^1 P_t(h)^{1-\epsilon}dh \\ &= \frac{c_t}{P_t^{-\epsilon}}P_t^{1-\epsilon} = c_t P_t\end{aligned}$$

where the last line was obtained by using (A2).

Proving (A1):

The consumption variety allocation problem consists on minimizing total expenditure for a given level of aggregate consumption:

$$\min_{c_t(h)} \int_0^1 P_t(h)c_t(h)dh \quad \text{s.t. } c_t = \left(\int_0^1 c_t(h)^{\frac{\epsilon-1}{\epsilon}} dh\right)^{\frac{\epsilon}{\epsilon-1}}$$

$\mathcal{L} = \int_0^1 P_t(h)c_t(h)dh + P_t\left[c_t - \left(\int_0^1 c_t(h)^{\frac{\epsilon-1}{\epsilon}} dh\right)^{\frac{\epsilon}{\epsilon-1}}\right]$

The FOC is:

$$\begin{aligned}[c_t(h)]: \quad & P_t(h) + P_t\left(\frac{-\epsilon}{\epsilon-1}\right)\overbrace{\left(\int_0^1 c_t(h)^{\frac{\epsilon-1}{\epsilon}} dh\right)^{\frac{\epsilon}{\epsilon-1}-1}}^{c_t^{\frac{1}{\epsilon}}}\left(\frac{\epsilon-1}{\epsilon}\right)c_t(h)^{\frac{\epsilon-1}{\epsilon}-1} = 0 \\ & \frac{P_t(h)}{P_t} = c_t^{\frac{1}{\epsilon}} c_t(h)^{\frac{-1}{\epsilon}} \\ & c_t(h) = \left(\frac{P_t(h)}{P_t}\right)^{-\epsilon} c_t\end{aligned}$$

Proving (A2):

Substitute the optimal demand $c_t(h)$ into the consumption aggregate:

$$\begin{aligned}
c_t &= \left(\int_0^1 c_t(h)^{\frac{\epsilon-1}{\epsilon}} dh\right)^{\frac{\epsilon}{\epsilon-1}} \\
&= \left(\int_0^1 \left[\left(\frac{P_t(h)}{P_t}\right)^{-\epsilon} c_t\right]^{\frac{\epsilon-1}{\epsilon}} dh\right)^{\frac{\epsilon}{\epsilon-1}} \\
&= c_t \left(\int_0^1 \left(\frac{P_t(h)}{P_t}\right)^{1-\epsilon} dh\right)^{\frac{\epsilon}{\epsilon-1}} \\
1 &= \int_0^1 \frac{P_t(h)^{1-\epsilon}}{P_t^{1-\epsilon}} dh \\
P_t &= \left(\int_0^1 P_t(h)^{1-\epsilon} dh\right)^{\frac{1}{1-\epsilon}}
\end{aligned}$$

← (end of detour)

Second, we divide both sides of the budget constraint by P_t to get a constraint in real terms (and substitute some of our initial definitions):

$$c_t + \frac{b_t}{R_t} = \frac{b_{t-1}}{\Pi_t} + \int_0^1 div_t(f)df + w_t n_t^s$$

Here $w_t = W_t/P_t$ is the real wage and we multiply the real bonds in $t-1$ for 1 $(\frac{B_{t-1}}{P_t}\frac{P_{t-1}}{P_{t-1}})$ to get the first term on the RHS.

Now we can set the lagrangian for the HH-UMP:

$$\mathcal{L}^{\mathcal{HH}} = \mathbb{E}_t \sum_{s=0}^{\infty} \beta^s \left\{ z_{t+s}\left(\frac{c_{t+s}^{1-\sigma}}{1-\sigma} - \frac{(n_{t+s}^{su})^{1+\psi}}{1+\psi}\right) + \lambda_{t+s}\left[\int_0^1 div_{t+s}(f)df + w_{t+s}n_{t+s}^{su} + \frac{b_{t-1+s}}{\Pi_{t+1t+s}} - c_{t+s} - \frac{b_{t+s}}{R_{t+s}}\right]\right\}$$

The FOCs are:

$$[c_t]: \quad c_t^{-\sigma} z_t - \lambda + t = 0 \;\Rightarrow\; \lambda_t = z_t c_t^{-\sigma}$$

$$[n_t^{su}]: \quad -(n_t^{su})^{\psi} z_t + \lambda_t w_t = 0 \;\Rightarrow\; w_t = \frac{(n_t^{su})^{\psi} z_t}{\lambda_t}$$

$$\begin{aligned}
[b_t]: \quad -\lambda_t \frac{1}{R_t} + \mathbb{E}_t \lambda_{t+1} \frac{1}{\Pi_{t+1}} \beta &= 0 \\
\lambda_t = \beta \mathbb{E}_t \left[\lambda_{t+1} \frac{R_t}{\Pi_{t+1}}\right]
\end{aligned}$$

From $[n_t^{su}]$, using the first $[c_t]$ we have,

$$w_t = \frac{(n_t^{su})^{\psi}}{c_t^{-\sigma}} \tag{2}$$

That is, the marginal utility of an extra wage dollar is equal to the utility cost of working. Furthermore, notice how the preference (demand) shock does not impact the labor supply decision.

Also, substituting the result of $[c_t]$ and the real rate r_t from the Fisher equation leads us to the Euler Equation:

$$z_t c_t^{-\sigma} = \beta \mathbb{E}_t \left[z_{t+1} c_{t+1}^{-\sigma} r_t\right] \tag{3}$$

Firms

The production sector will consist of (i) a representative Final Good Firm, that operates under perfect competition and bundles a set of intermediate goods (inputs), and (ii) a unit mass of intermediate goods firms (indexed by f) that produce input goods of different varieties using a linear production function in labor. The latter type of firms are each a monopolist of their own variety good f that are imperfect substitutes with other input varieties, and hence, they operate in a monopolistic competition environment.

Final Goods Firms

These firms are price takers and will determine the optimal demand of each input variety (they also determine the scale of their total production).

They have a technology of aggregation of intermediate goods given by: $y_t = \left[\int_0^1 y_t(f)^{\frac{\epsilon-1}{\epsilon}} df\right]^{\frac{\epsilon}{\epsilon-1}}$

The firm's PMP is:

$$\max_{y_t, y_t(h)} = \pi_t = P_t y_t - \int_0^1 P_t(f) y_t(f) df$$
$$s.t. \quad \left(\int_0^1 y_t(f)^{\frac{\epsilon-1}{\epsilon}} df\right)^{\frac{\epsilon}{\epsilon-1}} = y_t$$

The associated lagrangian is:

$$\mathcal{L} = \pi_t = P_t y_t - \int_0^1 P_t(f) y_t(f) df + \lambda_t^P \left[\left(\int_0^1 y_t(f)^{\frac{\epsilon-1}{\epsilon}} df\right)^{\frac{\epsilon}{\epsilon-1}} - y_t\right]$$

The FOCs are:

$$[y_t]: \quad P_t = \lambda_t^P$$

$$[y_t(f)]: \quad -P_t(f) + \lambda_t^P \left(\frac{\epsilon}{\epsilon-1}\right)\left(\int_0^1 y_t(f)^{\frac{\epsilon}{\epsilon-1}} df\right)^{\frac{\epsilon}{\epsilon-1}-1} \left(\frac{\epsilon-1}{\epsilon}\right) y_t(f)^{\frac{\epsilon-1}{\epsilon}-1} = 0$$
$$P_t(f) = \lambda_t^P y_t^{\frac{1}{\epsilon}} y_t(f)^{\frac{-1}{\epsilon}}$$
$$y_t(f) = \left(\frac{P_t(f)}{P_t}\right)^{-\epsilon} y_t$$

Where we substituted λ_t^P from $[y_t]$ to get the optimal input demand.

From the first FOC we get that the shadow price of the output is also the price level which is consistent with it being in units of the final consumption good.

The second FOC gives us the optimal demand for each variety in a similar fashion than the optimal consumption demand, i.e., the input variety demand increases with the aggregate output and decreases with its relative price.

As with the consumption variety allocation problem we can substitute the optimal input demand $y_t(f)$ in the definition of the aggregate output y_t and obtain $P_t = \left(\int_0^1 P_t(f) df\right)^{\frac{1}{1-\epsilon}}$

Intermediate Goods Firms

These firms are price setters. They will decide on their optimal inputs, i.e., the labor demand, and they will set the price of their good (input) variety f.

It is convenient to set each problem (demand and price setting) separately as one is static, the other is dynamic, and the latter also is affected by the price rigidity structure.

Since the PMP of the intermediate firm can be dynamic it is useful to define the present value of (nominal) dividends: $\mathbb{E}_t \sum_{s=0}^{\infty} \Lambda_{t,t+s} P_{t+s} div_{t+s}(f)$ where the firms discount (nominal) future flows using the stochastic discount factor $\Lambda_{t,t+s} = \beta^s \frac{\lambda_{t+s}}{\lambda}$ (with λ_{t+s} being the marginal utility of consumption in period $t+s$).

Optimal Input Demand decision

$$\max_{n_t^d} \mathbb{E}_t \Lambda_{t,t+s} P_{t+s} \overbrace{\left[\frac{P_{t+s}(f)}{P_{t+s}} y_{t+s}(f) - w_{t+s} n_{t+s}^d(f) \right]}^{div_{t+s}}$$
$$s.t. \quad y_t(f) = a_t n_t^d(f) \quad \forall t$$

The associated lagrangian is:

$$\mathcal{L}^{IntFF} = \mathbb{E}_t \sum_{s=0}^{\infty} \left\{ P_{t+s} \left(\frac{P_{t+s}(f)}{P_{t+s}} y_{t+s}(f) - w_{t+s} n_{t+s}^d(f) \right) + P_{t+s} mc_{t+s}(f) \Big(y_{t+s}(f) - a_{t+s} n_{t+s}^d \Big) \right\}$$

Where the lagrange multiplier is conveniently set in terms of the nominal marginal cost, or the prices level times the real marginal cost ($mc_{t+s}(f)$).[1] Also, notice that this problem in reality is static as the optimal decision of the firm does not affect the profits (or dividends) of future periods.

The FOC for this problem is:

$$\begin{aligned} [n_t^d(f)]: \quad & P_t(w_t - mc_t(f) a_t) = 0 \\ & w_t = mc_t(f) a_t \end{aligned} \tag{4}$$

It can be seen that the marginal costs are symmetric across firms as they don't depend on f in this linear production example. Thus, from here on we can drop the f index form the marginal costs.

We can also get the aggregate real marginal costs: $mc_t = \int_0^1 mc_t(f) df = \frac{w_t}{a_t}$. That is, the real marginal costs (aggregate) are equal to the wage in effective units (divided by the aggregate productivity).

Price Setting decision

Here we follow the time-contingent nominal rigidities setup of Calvo (1983) and Yun (1996). Out of the total mass of firms, a share $(1-\theta)$ are allowed to reset prices, the rest will keep the prices from the last period. In practice, this implies any firm will reset prices at t with a probability $1-\theta$ and set prices $P_t(f)$ as follows:

$$P_t(f) = \begin{cases} \tilde{P}_t(f) & \text{with Prob. } 1-\theta \\ P_{t-1}(f) & \text{with Prob. } \theta \end{cases}$$

Where $\tilde{P}_t$ is the optimal price at t.

With this setup, a firm that can reset its price will maximize its profits considering the probability of being stuck in every future period with the price they are choosing today. The probability of having to keep the same price s periods ahead is θ^s.

The PMP associated is:

$$\max_{P_t(f)} \mathbb{E}_t \sum_{s=0}^{\infty} \theta^s \Lambda_{t,t+s} P_{t+s} \left\{ \left(\frac{\tilde{P}_t(f)}{P_{t+s}} \right) y_{t+s}(f) - w_{t+s} n_{t+s}^d(f) \right\}$$
$$s.t. \quad y_{t+s}(f) = \left(\frac{\tilde{P}_t(f)}{P_{t+s}} \right)^{-\epsilon} y_{t+s}$$
$$y_{t+s}(f) = a_{t+s} n_{t+s}^d(f)$$

[1] Notice one could set the lagrange multiplier with any notation and as a number that could include time discounting and prices within, or not, like in this case.

Notice how in this case the objective function and the constraint have the same $P_t(f)$ at every period $t+s$. This reflects the consideration by the firm of the possibility of being stuck at the price they are choosing today for every period forward.

The lagrangian, after replacing $y_{t+s}(f)$ in the objective equation and other constraint, is:

$$\mathcal{L}^{\text{IntFF}} = \mathbb{E}_t \sum_{s=0}^{\infty} \theta^s \Lambda_{t,t+s} P_{t+s} \left\{ \left(\frac{\tilde{P}_t(f)}{P_{t+s}} \right)^{1-\epsilon} y_{t+s} - w_{t+s} n^d_{t+s}(f) + mc_{t+s} \left(a_{t+s} n^d_{t+s}(f) - \left(\frac{\tilde{P}_t(f)}{P_{t+s}} \right)^{-\epsilon} y_{t+s} \right) \right\}$$

Again, notice how the lagrange multiplier is $\theta^s \Lambda_{t,t+s} P_{t+s} mc_{t+s}$. This setup simplifies the algebra when taking FOCs (compared to just setting the multiplier as mc_{t+s} which could also be done).

The FOCs is:

$$[\tilde{P}_t(f)]: \quad \mathbb{E}_t \sum_{s=0}^{\infty} \theta^s \Lambda_{t,t+s} P^{\epsilon}_{t+s} y_{t+s} \left[(1-\epsilon)\tilde{P}_t(f)^{-\epsilon} + \epsilon P_{t+s} mc_{t+s} \tilde{P}_t(f)^{-\epsilon-1} \right] = 0$$

we can simplify this expression. First we multiply it times $\tilde{P}_t(f)^{\epsilon+1}$,

$$\mathbb{E}_t \sum_{s=0}^{\infty} \theta^s \Lambda_{t,t+s} P^{\epsilon}_{t+s} y_{t+s} \left[(1-\epsilon)\tilde{P}_t(f) + \epsilon P_{t+s} mc_{t+s} \right] = 0$$

Rearranging this expression,

$$\mathbb{E}_t \sum_{s=0}^{\infty} \theta^s \Lambda_{t,t+s} P^{\epsilon}_{t+s} y_{t+s} \tilde{P}_t(f) = \frac{\epsilon}{\epsilon-1} \mathbb{E}_t \sum_{s=0}^{\infty} \theta^s \Lambda_{t,t+s} P^{\epsilon+1}_{t+s} y_{t+s} mc_{t+s}$$

We divide by $P_t^{\epsilon+1}$ to arrive to a final expression:

$$\underbrace{\frac{\tilde{P}_t(f)}{P_t}}_{\tilde{p}_t} \underbrace{\mathbb{E}_t \sum_{s=0}^{\infty} \theta^s \Lambda_{t,t+s} \left(\frac{P_{t+s}}{P_t} \right)^{\epsilon} y_{t+s}}_{S_{1,t}} = \frac{\epsilon}{\epsilon-1} \underbrace{\mathbb{E}_t \sum_{s=0}^{\infty} \theta^s \Lambda_{t,t+s} \left(\frac{P_{t+s}}{P_t} \right)^{\epsilon+1} y_{t+s} mc_{t+s}}_{S_{2,t}}$$

$\tilde{p}_t$ is the relative optimal price of the intermediate firm f's good. Notice how it does not depend on f (it's the same for all intermediate firms).

$S_{1,t}$ and $S_{2,t}$ are auxiliary variables that although complicated can be set in a recursive form.

for $S_{1,t}$:

$$\begin{aligned} S_{1,t} &= \mathbb{E}_t \sum_{s=0}^{\infty} \theta^s \Lambda_{t,t+s} \left(\frac{P_{t+s}}{P_t} \right)^{\epsilon} y_{t+s} \\ &= y_t + \mathbb{E}_t \sum_{s=1}^{\infty} \theta^s \Lambda_{t,t+s} \left(\frac{P_{t+s}}{P_t} \right)^{\epsilon} y_{t+s} \end{aligned}$$

re-setting the summation so that it starts at $t=0$ and multiplying the second RHS term by $1 = \frac{P_{t+1}}{P_{t+1}}$:

$$S_{1,t} = y_t + \mathbb{E}_t \sum_{s=0}^{\infty} \theta^{s+1} \Lambda_{t,t+s+1} \left(\frac{P_{t+s+1}}{P_{t+1}} \frac{P_{t+1}}{P_t} \right)^{\epsilon} y_{t+s+1}$$

Now we use that $\beta\frac{\lambda_{t+1}}{\lambda_t}\Pi_{t+1}^{-1}\Lambda_{t+1,t+1+s} = \Lambda_{t,t+s+1}$,

$$S_{1,t} = y_t + \mathbb{E}_t\sum_{s=0}^{\infty}\theta^{s+1}\beta\frac{\lambda_{t+1}}{\lambda_t}\Pi_{t+1}^{-1}\Lambda_{t+1,t+1+s}\left(\frac{P_{t+s+1}}{P_{t+1}}\Pi_{t+1}\right)^{\epsilon}y_{t+s+1}$$

$$S_{1,t} = y_t + \mathbb{E}_t\theta\beta\frac{\lambda_{t+1}}{\lambda_t}\Pi_{t+1}^{\epsilon-1}\mathbb{E}_t\sum_{s=0}^{\infty}\theta^{s}\Lambda_{t+1,t+1+s}\left(\frac{P_{t+1+s}}{P_{t+1}}\right)^{\epsilon}y_{t+1+s}$$

$$S_{1,t} = y_t + \mathbb{E}_t\theta\beta\frac{\lambda_{t+1}}{\lambda_t}\Pi_{t+1}^{\epsilon-1}S_{1,t+1}$$

Similarly for $S_{2,t}$

$$\begin{aligned}
S_{2,t} &= \mathbb{E}_t\sum_{s=0}^{\infty}\theta^{s}\Lambda_{t,t+s}\left(\frac{P_{t+s}}{P_t}\right)^{\epsilon+1}y_{t+s}mc_{t+s} \\
&= y_t mc_t + \mathbb{E}_t\sum_{s=1}^{\infty}\theta^{s}\Lambda_{t,t+s}\left(\frac{P_{t+s}}{P_t}\right)^{\epsilon+1}y_{t+s}mc_{t+s} \\
&= y_t mc_t + \mathbb{E}_t\sum_{s=0}^{\infty}\theta^{s+1}\Lambda_{t,t+s+1}\left(\frac{P_{t+s+1}}{P_t}\frac{P_{t+1}}{P_{t+1}}\right)^{\epsilon+1}y_{t+s+1}mc_{t+s+1} \\
&= y_t mc_t + \theta\mathbb{E}_t\sum_{s=0}^{\infty}\theta^{s}\beta\tfrac{\lambda_{t+1}}{\lambda_t}\Pi_{t+1}^{-1}\Lambda_{t+1,t+1+s}\left(\frac{P_{t+s+1}}{P_{t+1}}\Pi_{t+1}\right)^{\epsilon+1}y_{t+s+1}mc_{t+s+1} \\
&= y_t mc_t + \mathbb{E}_t\theta\beta\tfrac{\lambda_{t+1}}{\lambda_t}\Pi_{t+1}^{\epsilon}\mathbb{E}_t\sum_{s=0}^{\infty}\theta^{s}\Lambda_{t+1,t+1+s}\left(\frac{P_{t+1+s}}{P_{t+1}}\right)^{\epsilon+1}y_{t+1+s}mc_{t+1+s} \\
S_{2,t} &= y_t mc_t + \mathbb{E}_t\theta\beta\tfrac{\lambda_{t+1}}{\lambda_t}\Pi_{t+1}^{\epsilon}S_{2,t+1}
\end{aligned}$$

Thus, the new equations are:

$$\tilde{p}_t = \frac{\epsilon}{\epsilon-1}\frac{S_{2,t}}{S_{1,t}}$$
$$S_{1,t} = y_t + \mathbb{E}_t\theta\beta\frac{\lambda_{t+1}}{\lambda_t}\Pi_{t+1}^{\epsilon-1}S_{1,t+1}$$
$$S_{2,t} = y_t mc_t + \mathbb{E}_t\theta\beta\tfrac{\lambda_{t+1}}{\lambda_t}\Pi_{t+1}^{\epsilon}S_{2,t+1}$$

Before proceeding, it's useful to simplify these equations so that we don't have to add λ_{t+1}, λ_t to our final system of equations; for that, re-state these equations in terms of the variables $s_{1,t} = \lambda_t S_{1,t}$, and $s_{2,t} = \lambda_t S_{2,t}$:

$$\tilde{p}_t = \frac{\epsilon}{\epsilon-1}\frac{s_{2,t}}{s_{1,t}} \tag{5}$$
$$s_{1,t} = y_t u_{c,t} + \theta\beta\mathbb{E}_t\Pi_{t+1}^{\epsilon-1}s_{1,t+1} \tag{6}$$
$$s_{2,t} = y_t mc_t u_{c,t} + \theta\beta\mathbb{E}_t\Pi_{t+1}^{\epsilon}s_{2,t+1} \tag{7}$$

where $u_{c,t} = \lambda_t = \frac{\partial u(c_t,n_t^s,z_t)}{\partial c_t}$ is the marginal utility of consumption.

Now, we can also get the law of motion of the optimal price (while accounting for the nominal rigidities):

From the aggregate price index:

$$1 = \int_0^1\left(\frac{P_t(f)}{P_t}\right)^{1-\epsilon}df$$

Given the nominal rigidity setup, we can split the integral into that of those re-setting prices and those who cannot:

$$1 = \int_{\text{Optim}}\left(\frac{P_t(f)}{P_t}\right)^{1-\epsilon}df + \int_{\text{NonOptim}}\left(\frac{P_t(f)}{P_t}\right)^{1-\epsilon}df$$

Now we can replace the choice variable (firm's price $P_t(f)$) by its optimal value (which does not depend on f) and factor it and also replace the price of firms that are stuck in the previous price level. Also, notice we can evaluate both integrals from 0 to 1 since we are factoring out their relative masses:

$$
\begin{aligned}
1 &= (1-\theta)\left(\frac{\tilde{P}_t}{P_t}\right)^{1-\epsilon} + \theta\int_0^1 \left(\frac{P_{t-1}(f)}{P_t}\right)^{1-\epsilon} df \\
1 &= (1-\theta)\left(\tilde{p}_t\right)^{1-\epsilon} + \theta\int_0^1 \left(\frac{P_{t-1}(f)}{P_t}\frac{P_{t-1}}{P_{t-1}}\right)^{1-\epsilon} df \\
1 &= (1-\theta)\left(\tilde{p}_t\right)^{1-\epsilon} + \theta\left(\frac{P_{t-1}}{P_t}\right)^{1-\epsilon} \cancelto{1}{\int_0^1 \left(\frac{P_{t-1}(f)}{P_{t-1}}\right)^{1-\epsilon}} df
\end{aligned}
$$

After simplifications, we characterized the law of motion of the optimal relative price (i.e. how the new price $\tilde{p}_t$ depends on past aggregate prices within the inflation term),

$$
1 = (1-\theta)\tilde{p}_t^{1-\epsilon} + \theta\Pi_t^{\epsilon-1} \tag{8}
$$

Market Clearing

Bonds

The usual zero net supply condition should hold (supply equals demand). Together with a closed economy setup with symmetric agents it implies:

$$
B_t = 0
$$

Labor

Similarly, we can equal supply and demand in this market:

$$
n_t^{sup} = \int_0^1 n_t^d(f)df = n_t
$$

Goods

Aggregate inputs spending: We can simplify the aggregate spending in intermediate goods with the results we obtained (optimal demand and associated prices).

$$
\begin{aligned}
\int_0^1 y_t(f)P_t(f)df &= \int_0^1 \left(\frac{P_t(f)}{P_t}\right)^{-\epsilon} y_t P_t(f)df \\
&= P_t y_t \cancelto{1}{\int_0^1 \left(\frac{P_t(f)}{P_t}\right)^{1-\epsilon}} df \\
&= P_t y_t
\end{aligned}
$$

Here in the first line we substituted the optimal demand and in the second we replace the integral by one given the expression for the prices. The resulting equation states that the aggregate spending must equal the aggregate price level times the total output.

Aggregate dividends: similarly, we can simplify the dividends (given the optimal demand and prices).

$$\begin{aligned} div_t = \int_0^1 div_t(f)df &= \int_0^1 \frac{P_t(f)}{P_t} y_t(f)df - \int_0^1 w_t n_t^d(f)df \\ &= \int_0^1 \frac{P_t(f)}{P_t}\left(\frac{P_t(f)}{P_t}\right)^{-\epsilon} y_t df - w_t \underbrace{\int_0^1 n_t^d(f)df}_{n_t} \\ &= y_t \underbrace{\int_0^1 \left(\frac{P_t(f)}{P_t}\right)^{1-\epsilon}}_{\to 1} df - w_t n_t \\ div_t &= y_t - w_t n_t \end{aligned} \tag{9}$$

From this equation we can tell that real aggregate profits will equal the total output minus the product of the real wage times aggregate labor.

Now, we substitute these results into the budget constraint of the households to obtain a simplified expression:

$$\begin{aligned} \int_0^1 \frac{P_t(f)}{P_t} c_t(h)df + Q_t b_t &= b_{t-1}\Pi_t^{-1} + w_t n_t^{sup} + \int_0^1 div_t(f)df \\ \frac{P_t}{P_t} c_t + Q_t \cancelto{0}{b_t} &= \cancelto{0}{b_{t-1}}\Pi_t^{-1} + \cancel{w_t n_t} + y_t - \cancel{w_t n_t} \\ c_t &= y_t \end{aligned} \tag{10}$$

From here we can see that the aggregate demand of final goods (c_t) equals the aggregate output (y_t).

Aggregate supply

This one is a bit different given the presence of nominal rigidities on the supply side of the economy and thus of inefficiencies:

Without inefficiencies we have $y_t = a_t n_t$, i.e., aggregate output equals aggregate supply. We will see this result changes with the nominal distortion.

Let us obtain the total output from each side. First, from the aggregation of the optimal input demands:

$$\begin{aligned} y_t^{total} &= \int_0^1 y_t(f)df \\ &= \int_0^1 \left(\frac{P_t(f)}{P_t}\right)^{-\epsilon} y_t df \\ &= y_t \underbrace{\int_0^1 \left(\frac{P_t(f)}{P_t}\right)^{-\epsilon} df}_{\text{price dispersion}=p_t^*} \end{aligned}$$

Here $p_t^* \neq 1$ is the price dispersion.

Now, aggregating the input production functions:

$$\begin{aligned} y_t^{total} &= \int_0^1 y_t(f)df \\ &= \int_0^1 a_t n_t^d(f)df \\ &= a_t n_t \end{aligned}$$

Equating both results,

$$p_t^* y_t = a_t n_t \tag{11}$$

Unlike the demand side, here we have that aggregate supply is not equal to the total output, and actually, is lower, as $p_t^* \leq 1$. Intuitively, this makes sense, as a lower than efficient supply is consistent with a price that is set with a mark-up (higher than competitive prices).

Analogously to the optimal price, we can get an expression for the dynamics of the price dispersion:

$$\begin{aligned}
p_t^* &= \int_0^1 \left(\frac{P_t(f)}{P_t}\right)^{-\epsilon} df \\
&= \int_{\text{Opt}} \left(\frac{P_t(f)}{P_t}\right)^{-\epsilon} df + \int_{\text{NonOpt}} \left(\frac{P_t(f)}{P_t}\right)^{-\epsilon} df \\
&= (1-\theta)\tilde{p}_t^{-\epsilon} \overbrace{\int_0^1 1 df}^{1} + \theta \int_0^1 \left(\frac{P_{t-1}(f)}{P_{t-1}} \frac{P_{t-1}}{P_t}\right)^{-\epsilon} df \\
&= (1-\theta)\tilde{p}_t^{-\epsilon} + \theta \Pi_t^{\epsilon} \underbrace{\int_0^1 \left(\frac{P_{t-1}(f)}{P_{t-1}}\right)^{-\epsilon} df}_{p_{t-1}^*}
\end{aligned}$$

$$p_t^* = (1-\theta)\tilde{p}_t^{-\epsilon} + \theta \Pi_t^{\epsilon} p_{t-1}^* \tag{12}$$

Monetary Policy

We assume a standard feedback rule (Taylor rule) that depends on the steady state interest rate value, the inflation gap (relative to the steady state or a target), and the output gap.

$$R_t = R \left(\frac{\Pi_t}{\Pi^*}\right)^{\phi_\pi} \left(\frac{y_t}{y}\right)^{\phi_y} e^{\nu_t} \tag{13}$$

Exogenous Processes

Finally, we assume the three exogenous variables (preferences shock, TFP, Monetary policy shock) follow an AR(1) process, each affected by a normally distributed stochastic shock.

Preferences:

$$\log z_t = \rho_z \log z_{t-1} + \epsilon_{z,t} \tag{14}$$

TPF (productivity):

$$\log a_t = \rho_z \log a_{t-1} + \epsilon_{a,t} \tag{15}$$

Monetary policy shock:

$$\nu_t = \rho_z \nu_{t-1} + \epsilon_{\nu,t} \tag{16}$$

with,

$$\begin{pmatrix} \epsilon_{z,t} \\ \epsilon_{a,t} \\ \epsilon_{\nu,t} \end{pmatrix} \sim N \left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma_z^2 & 0 & 0 \\ 0 & \sigma_a^2 & 0 \\ 0 & 0 & \sigma_\nu^2 \end{pmatrix} \right)$$

Final set of equations (model summary)

At this point we have a system of equations we can use for solving the model:

Table 1: Model summary

Equation	Description
1. $r_t = \mathbb{E}_t \frac{R_t}{\Pi_{t+1}}$	Fisher Equation (real interest rate)
2. $w_t = \frac{n_t^{\psi}}{c_t^{-\sigma}}$	Intratemporal Euler Equation (optimal wage)
3. $z_t c_t^{-\sigma} = \beta \mathbb{E}_t \left[z_{t+1} c_{t+1}^{-\sigma} r_t \right]$	Euler Equation
4. $w_t = mc_t a_t$	Optimal labor demand and wage
5. $\tilde{p}_t = \frac{\epsilon}{\epsilon - 1} \frac{s_{2,t}}{s_{1,t}}$	Optimal price setting
6. $s_{1,t} = y_t u_{c,t} + \theta \beta \mathbb{E}_t \Pi_{t+1}^{\epsilon-1} s_{1,t+1}$	Optimal price setting auxiliary recursion 1
7. $s_{2,t} = y_t mc_t u_{c,t} + \theta \beta \mathbb{E}_t \Pi_{t+1}^{\epsilon} s_{2,t+1}$	Optimal price setting auxiliary recursion 2
8. $1 = (1-\theta) \tilde{p}_t^{1-\epsilon} + \theta \Pi_t^{\epsilon - 1}$	Law of motion of optimal reset prices
9. $div_t = y_t - w_t n_t$	Aggregate dividends
10. $c_t = y_t$	Aggregate demand
11. $p_t^* y_t = a_t n_t$	Aggregate supply
12. $p_t^* = (1-\theta) \tilde{p}_t^{-\epsilon} + \theta \Pi_t^{\epsilon} p_{t-1}^*$	Law of motion of price distortion
13. $R_t = R \left(\frac{\Pi_t}{\Pi^*} \right)^{\phi_\pi} \left(\frac{y_t}{y} \right)^{\phi_y} e^{\nu_t}$	Taylor Rule
14. $\log z_t = \rho_z \log z_{t-1} + \epsilon_{z,t}$	Preferences shock
15. $\log a_t = \rho_z \log a_{t-1} + \epsilon_{a,t}$	Total factor productivity (technology shock)
16. $\nu_t = \rho_z \nu_{t-1} + \epsilon_{\nu,t}$	Monetary policy shock

We use these system to solve for the equilibrium value of these variables:

Variables: $r_t, R_t, \Pi_t, w_t, n_t, c_t, y_t, mc_t, \tilde{p}, s_{1,t}, s_{2,t}, div_t, p_t^*, z_t, a_t, \nu_t$ (total: 16)

Notice that: $u_{c,t} = c_t^{-\sigma}$ in this case; and we are directly using that $n_t^{su} = n_t$ in equilibrium, as well as $mc_t(f) = mc_t$

In addition, for reporting purposes we consider the following auxiliary variables and associated equations:

Table 2: Additional auxiliary variables

Equation	Description
17. $Q_t = \frac{1}{R_t}$	Price of bonds
18. $\hat{y}_t = \log y_t - \log y$	definition of log output (deviation from steady-state)
19. $\hat{w}_t = \log w_t - \log w$	definition of log real wage (deviation from steady-state)
20. $\hat{n}_t = \log n_t - \log n$	definition of log hours worked (deviation from steady-state)
21. $\hat{\Pi}_t^{an} = 4(\log \Pi_t - \log \Pi)$	definition of log annualized inflation (deviation from steady-state)
22. $\hat{R}_t^{an} = 4(\log R_t - \log R)$	definition of log annualized nominal interest rate (deviation from steady-state)
23. $\hat{r}_t^{an} = 4(\log r_t - \log r)$	definition of log annualized real interest rate (deviation from steady-state)
24. $\hat{mc}_t = \log mc_t - \log mc$	definition of log marginal costs (deviation from steady-state)
25. $\hat{a}_t = \log a_t - \log a$	definition of log total factor productivity (deviation from steady-state)
26. $\hat{z}_t = \log z_t - \log z$	definition of log of preferences shifter (deviation from steady-state)

Parameters

We will use the following calibrated parameter values, mostly from Gali (2015):

Table 3: Parameter Values

Parameter	Value	Description
β	0.990	discount factor
ρ_a	0.900	autocorrelation technology process
ρ_ν	0.500	autocorrelation monetary policy process
ρ_z	0.500	autocorrelation preference shock
σ	1.000	inverse elasticity of intertemporal substitution
φ	5.000	inverse Frisch elasticity
ϕ_π	1.500	inflation feedback Taylor Rule
ϕ_y	0.125	output feedback Taylor Rule
ϵ	9.000	demand elasticity
θ	0.750	Calvo parameter
π^*	1.005	inflation target

Steady State

This version of the model allows for an analytical solution of the (non-stochastic) steady-state.

The exogenous processes equations will have the following steady-state values:

$$z_t = 1 \qquad a_t = 1 \qquad \nu_t = 0$$

From the monetary policy rule (eq. (15)) we get that the steady-state value of inflation is given by its policy target:

$$\Pi = \Pi^*$$

From the law of motion of prices (12)

$$\tilde{p} = \left(\frac{1 - \theta \Pi^{\epsilon - 1}}{1 - \theta} \right)^{\frac{1}{1-\epsilon}}$$

The algebra simplifies greatly with long-run price stability (zero inflation), e.g., $\tilde{p}$ would be one.

Now we simplify the optimal pricing conditions (optimal pricing and two auxiliar conditions). First, from the auxiliar conditions (6) and (7) we obtain an expression for $\frac{s_1}{s_2}$, then we replace it in equation (5), and solve for the steady-state marginal cost:

$$mc = \frac{\epsilon - 1}{\epsilon} \left(\frac{1 - \beta\theta\Pi^{\epsilon}}{1 - \beta\theta\Pi^{\epsilon - 1}} \right) \tilde{p}$$

From the law of motion of the price dispersion, we get:

$$p^* = \frac{1 - \theta}{1 - \theta\Pi^{\epsilon}} \tilde{p}^{-\epsilon}$$

Now, from (3), (1), (17),

$$r = 1/\beta \qquad R = r\Pi \qquad Q = 1/R$$

From (4) we obtain the steady-state wages:

$$w = mc \cdot a$$

For labor, in (2): replace (10), and substitute y from (11): $w = n^{\psi} c^{\sigma} = n^{\psi} y^{\sigma} = n^{\psi} \left(\frac{an}{p^*} \right)^{\sigma} = n^{\psi + \sigma} \left(\frac{a}{p^*} \right)^{\sigma}$, then:

$$n = \left(w \left(\frac{p^*}{a} \right)^{\sigma} \right)^{\frac{1}{\psi + \sigma}}$$

For the output we can now use (11), for consumption (10), and for the dividends (9):

$$y = an/p^* \qquad c = y \qquad div = y - w \cdot n$$

On the other hand, from (6), (7)

$$s_1 = \frac{y c^{\sigma}}{1 - \theta\beta\Pi^{\epsilon - 1}} \qquad s_2 = \frac{y \cdot mc \cdot c^{\sigma}}{1 - \theta\beta\Pi^{\epsilon}}$$

Finally, all "hatted" variables have a zero steady-state by definition: if $\hat{x}_t = x_t - x \ \Rightarrow \hat{x} = x - x = 0$