Short Note on the Method of Undetermined Coefficients (MUC) in the RBC Model

Our log-linearized model is:

$$k_{t+1} = \lambda_1 k_t + \lambda_2 a_t + (1 - \lambda_1 - \lambda_2)\, c_t \tag{27a}$$

$$\mathbb{E}_t(c_{t+1} - c_t) = \sigma\lambda_3\, \mathbb{E}_t(a_{t+1} - k_{t+1}) \tag{27b}$$

$$a_t = \phi\, a_{t-1} + \varepsilon_t \tag{27c}$$

This system is much simpler than the model in the original variables, but is not "solved" yet. The *solution* we are looking for are equations for the economic variables *decided by the agents* (c_t, k_{t+1}) as a function of the *states of the economy* (a_t, k_t).

To get that solution we use the MUC. Let's start by **guessing** that our solutions for c_t and k_{t+1} are *linear* functions of the states:

$$c_t = \eta_{ck}\, k_t + \eta_{ca}\, a_t \tag{Sol/c}$$

$$k_{t+1} = \eta_{kk}\, k_t + \eta_{ka}\, a_t \tag{Sol/k}$$

The MUC consists of replacing these guesses in the model and obtaining expressions of the form:

$$\underbrace{\eta_{ck}}_{\text{unknown}}\; k_t + \underbrace{\eta_{ca}}_{\text{unknown}}\; a_t = \text{expression}_1 \cdot k_t + \text{expression}_2 \cdot a_t$$

then the solution for the coefficients would be $\eta_{ck} = \text{expression}_1$, $\eta_{ca} = \text{expression}_2$. The idea is to find the coefficients of the solution that are *compatible* with our model (27).

Let's see how to get η_{ck}, η_{ca}, η_{kk}, η_{ka}:

Step 1. Replace (Sol/c) in equation (27a):

$$\begin{aligned} k_{t+1} &= \lambda_1 k_t + \lambda_2 a_t + (1 - \lambda_1 - \lambda_2)\overbrace{\left(\eta_{ck}\, k_t + \eta_{ca}\, a_t\right)}^{c_t} \\ &= \underbrace{\left[\lambda_1 + (1 - \lambda_1 - \lambda_2)\eta_{ck}\right]}_{\eta_{kk}}\; k_t \;+\; \underbrace{\left[\lambda_2 + (1 - \lambda_1 - \lambda_2)\eta_{ca}\right]}_{\eta_{ka}}\; a_t \\ k_{t+1} &= \eta_{kk} k_t + \eta_{ka} a_t \end{aligned} \tag{29}$$

Matching with (Sol/k) we immediately obtain:

$$\boxed{\eta_{kk} = \lambda_1 + (1 - \lambda_1 - \lambda_2)\,\eta_{ck}} \qquad \boxed{\eta_{ka} = \lambda_2 + (1 - \lambda_1 - \lambda_2)\,\eta_{ca}}$$

We have already found expressions for η_{kk} and η_{ka}, provided that we find η_{ck} and η_{ca}.

Step 2. Substitute (Sol/c) into equation (27b); substitute both c_{t+1} and c_t:

$$
\begin{aligned}
\mathbb{E}_t(c_{t+1} - c_t) &= \sigma\lambda_3\,\mathbb{E}_t(a_{t+1} - k_{t+1}) \\
\mathbb{E}_t\left((\eta_{ck}k_{t+1} + \eta_{ca}a_{t+1}) - (\eta_{ck}k_t + \eta_{ca}a_t)\right) &= \sigma\lambda_3\,\mathbb{E}_t a_{t+1} - \sigma\lambda_3\,k_{t+1} \\
\eta_{ck}(k_{t+1} - k_t) + \eta_{ca}\,\mathbb{E}_t(a_{t+1} - a_t) &= \sigma\lambda_3\,\mathbb{E}_t a_{t+1} - \sigma\lambda_3\,k_{t+1} \qquad (30)
\end{aligned}
$$

Substitute (29) ($k_{t+1} = \eta_{kk}\,k_t + \eta_{ka}\,a_t$) and $\mathbb{E}_t a_{t+1} = \phi\,a_t$ into (30):

$$
\begin{aligned}
\eta_{ck}\big(\eta_{kk}\,k_t + \eta_{ka}\,a_t - k_t\big) + \eta_{ca}\big(\phi\,a_t - a_t\big) &= \sigma\lambda_3\,\phi\,a_t - \sigma\lambda_3\big(\eta_{kk}\,k_t + \eta_{ka}\,a_t\big) \\
\eta_{ck}(\eta_{kk} - 1)\,k_t + \big[\eta_{ck}\eta_{ka} + \eta_{ca}(\phi - 1)\big]a_t &= -\sigma\lambda_3\eta_{kk}\,k_t + \sigma\lambda_3(\phi - \eta_{ka})\,a_t \qquad (31)
\end{aligned}
$$

Replacing $\eta_{kk} = \lambda_1 + (1 - \lambda_1 - \lambda_2)\eta_{ck}$ and $\eta_{ka} = \lambda_2 + (1 - \lambda_1 - \lambda_2)\eta_{ca}$ from (29) into (31), and applying the core idea of MUC (equate coefficients of k_t and a_t on both sides):

$$
\eta_{ck}\big(\eta_{kk} - 1\big) = -\sigma\lambda_3\,\eta_{kk} \qquad \text{(MUC-1)}
$$

$$
\eta_{ck}\,\eta_{ka} + \eta_{ca}(\phi - 1) = \sigma\lambda_3(\phi - \eta_{ka}) \qquad \text{(MUC-2)}
$$

From (MUC-1) we can get η_{kk}, then replace in (MUC-2) to get η_{ca}. The algebra involves solving a quadratic:

$$
Q_2\eta_{ck}^2 + Q_1\eta_{ck} + Q_0 = 0
$$

where

$$
\begin{aligned}
&Q_0 = \sigma\lambda_3\lambda_1, \\
&Q_1 = \lambda_1 - 1 + \sigma\lambda_3\,(1 - \lambda_1 - \lambda_2), \\
&Q_2 = 1 - \lambda_1 - \lambda_2.
\end{aligned}
$$

This is solved as a standard quadratic equation. Taking the economically relevant root (positive η_{ck}), the solutions are:

$$
\eta_{ck} = \frac{-Q_1 - \sqrt{Q_1^2 - 4\,Q_2\,Q_0}}{2\,Q_2} \qquad \text{(S1)}
$$

$$
\eta_{ca} = \frac{-\eta_{ck}\lambda_2 + \sigma\lambda_3(\phi - \lambda_2)}{\phi - 1 + (1 - \lambda_1 - \lambda_2)(\eta_{ck} + \sigma\lambda_3)} \qquad \text{(S2)}
$$

$$
\eta_{ka} = \lambda_2 + (1 - \lambda_1 - \lambda_2)\,\eta_{ca} \qquad \text{(S3)}
$$

$$
\eta_{kk} = \lambda_1 + (1 - \lambda_1 - \lambda_2)\,\eta_{ck} \qquad \text{(S4)}
$$