A Guide to Log-Linearization

Econ - 5322 UTD

This handout provides a self-contained treatment of log-linearization techniques commonly used in macroeconomics. We begin with the single variable case and build up to functions of multiple variables and composite functions, deriving the standard "tricks" (power, sum, product, ratio) along the way. The final section addresses the less standard—but important—case in which one or more variables have a zero or negative steady-state value, requiring a modified normalization.[1]

1 Introduction

Dynamic stochastic general equilibrium (DSGE) models are typically solved by approximating the equilibrium conditions around a deterministic steady state. The first-order (linear) approximation expressed in terms of percentage deviations from steady state—commonly called *log-linearization*—is the workhorse technique. This note collects the key results and derivations in one place.

Throughout, we use the convention that uppercase letters denote the level of a variable (e.g., X_t), a bar denotes the trend or steady-state ($\bar{X}$), which for simplicity we assume constant over time, and the corresponding lowercase letter denotes the percentage deviation from steady state:

$$x_t \equiv \frac{X_t - \bar{X}}{\bar{X}}.$$

As we show below, this is equivalent (up to first order) to the log-deviation $\ln X_t - \ln \bar{X}$.

2 Functions of a single variable

Consider a relationship

$$Z_t = f(X_t),$$

where f is continuously differentiable. We wish to approximate this around the steady state $(\bar{Z}, \bar{X})$ satisfying $\bar{Z} = f(\bar{X})$. A first-order Taylor expansion gives

$$Z_t = f(\bar{X}) + f'(\bar{X})\,(X_t - \bar{X}) + \mathcal{O}\big((X_t - \bar{X})^2\big).$$

Dropping the second-order residual and assuming $\bar{Z} \neq 0$ and $\bar{X} \neq 0$, we can divide through and rearrange to obtain

$$z_t = \eta_f\, x_t, \tag{1}$$

where

$$\eta_f \equiv \frac{\bar{X}\, f'(\bar{X})}{f(\bar{X})}$$

[1] This handout borrows extensively from similar notes posted by other economists. The reason it is re-written here is because it gathers useful info from several of such sources, and because the sources themselves may no longer be publicly available, so it is not possible to share the materials by only providing a link.

is the **elasticity** of f evaluated at $\bar{X}$. The intuition is straightforward: the percentage change in Z_t around the steady state equals the elasticity of f times the percentage change in X_t.

2.1 The Power Rule

A particularly useful special case arises when f is a power function, $Z_t = X_t^{\omega}$. Computing the elasticity gives $\eta_f = \omega$ regardless of $\bar{X}$ (verify this). Hence

$$z_t = \omega\, x_t. \tag{2}$$

This result is used quite frequently and also be known as the ”power trick”.

3 Functions of Several Variables

The same logic extends naturally. Suppose

$$Z_t = f(X_t, Y_t),$$

with steady state $\bar{Z} = f(\bar{X}, \bar{Y})$. A first-order Taylor expansion in both arguments yields

$$z_t \;=\; \eta_f^x\, x_t \;+\; \eta_f^y\, y_t, \tag{3}$$

where the partial elasticities are

$$\eta_f^x = \frac{\bar{X}\, f_1(\bar{X}, \bar{Y})}{f(\bar{X}, \bar{Y})}, \qquad \eta_f^y = \frac{\bar{Y}\, f_2(\bar{X}, \bar{Y})}{f(\bar{X}, \bar{Y})}.$$

Here f_1 and f_2 denote the partial derivatives with respect to the first and second arguments, respectively. The result in (3) is the foundation for all of the “tricks” below.

3.1 The Sum Rule

Let $Z_t = X_t + Y_t$. Since $f_1 = f_2 = 1$ and $\bar{Z} = \bar{X} + \bar{Y}$, the partial elasticities are simply the steady-state shares:

$$z_t = \frac{\bar{X}}{\bar{X} + \bar{Y}}\, x_t \;+\; \frac{\bar{Y}}{\bar{X} + \bar{Y}}\, y_t. \tag{4}$$

In other words, the percentage deviation of a sum is the share-weighted average of the percentage deviations of its components.

3.2 The Product Rule

Let $Z_t = X_t Y_t$. Then $f_1 = Y_t$, $f_2 = X_t$, and a quick computation shows both partial elasticities equal one:

$$z_t = x_t + y_t. \tag{5}$$

3.3 The Ratio Rule

Let $Z_t = X_t/Y_t$ (with $\bar{Y} \neq 0$). By analogous reasoning:

$$z_t = x_t - y_t. \tag{6}$$

4 Composite Functions

Now suppose

$$Z_t = f(Y_t), \qquad Y_t = g(X_t),$$

so that $Z_t = f(g(X_t)) \equiv h(X_t)$. By the chain rule, $h'(X_t) = f'(Y_t)\, g'(X_t)$. Applying the elasticity formula from Section 2 to h and simplifying, one obtains

$$z_t = \eta_f^y \; \cdot \; \eta_g^x \; \cdot \; x_t, \tag{7}$$

where η_f^y is the elasticity of f at $\bar{Y}$ and η_g^x is the elasticity of g at $\bar{X}$. In words, the percentage deviation of Z_t equals the product of the two elasticities times the percentage deviation of X_t. Equivalently, one can think of this in two steps: first compute the percentage deviation of Y_t (which is $\eta_g^x\, x_t$), then multiply by η_f^y.

Example 1 (CES Aggregator)**.** *Consider the CES production function from Farmer:*

$$Y_t = \left(a\, X_t^\theta + (1-a)\, Z_t^\theta\right)^{\gamma/\theta}.$$

We can log-linearize this by applying the rules in sequence. Define the inner term $W_t \equiv a\, X_t^\theta + (1-a)\, Z_t^\theta$, so $Y_t = W_t^{\gamma/\theta}$.

Step 1 (Power rule on the outer function):

$$y_t = \frac{\gamma}{\theta}\, w_t.$$

Step 2 (Sum rule on W_t):

$$w_t = \frac{a\bar{X}^\theta}{a\bar{X}^\theta + (1-a)\bar{Z}^\theta} \times \text{(p.d. of } aX_t^\theta\text{)} + \frac{(1-a)\bar{Z}^\theta}{a\bar{X}^\theta + (1-a)\bar{Z}^\theta} \times \text{(p.d. of } (1-a)Z_t^\theta\text{)}.$$

Step 3 (Power rule on each term):

$$\text{p.d. of } aX_t^\theta = \theta\, x_t, \qquad \text{p.d. of } (1-a)Z_t^\theta = \theta\, z_t.$$

Collecting terms:

$$y_t = \gamma\, \frac{a\bar{X}^\theta\, x_t + (1-a)\bar{Z}^\theta\, z_t}{\bar{Y}^{\theta/\gamma}}.$$

5 Log-Deviations and Percentage Deviations

An important observation ties everything together. Apply a first-order Taylor expansion to $\ln X_t$ around $\bar{X}$:

$$\ln X_t \;\approx\; \ln \bar{X} + \frac{1}{\bar{X}}(X_t - \bar{X}) = \ln \bar{X} + x_t.$$

Rearranging,

$$x_t \;\approx\; \ln X_t - \ln \bar{X}. \tag{8}$$

That is, percentage deviations and log-deviations are equivalent up to first order.

This equivalence provides a useful shortcut for expressions that become linear in logs. For instance, with a Cobb–Douglas production function $Y_t = A_t K_t^{\alpha} L_t^{1-\alpha}$, taking logs on both sides and subtracting the steady-state version immediately yields

$$y_t = a_t + \alpha\, k_t + (1-\alpha)\, l_t.$$

There is no need to go through Taylor expansions and elasticities when the relationship is already log-linear.

6 Variables with Zero or Negative Steady-State Values

All of the preceding derivations assumed $\bar{X} \neq 0$ for every variable involved, because we divided by the steady-state value when defining percentage deviations. When a variable has a zero (or negative) steady-state value, the standard definition breaks down—we cannot compute $x_t = (X_t - \bar{X})/\bar{X}$ if $\bar{X} = 0$, nor can we take $\ln \bar{X}$ if $\bar{X} \leq 0$.

The solution is to **normalize by a different quantity**. In particular, instead of defining the hat variable as a percentage deviation, we define

$$\hat{x}_t \;\equiv\; \frac{X_t - \bar{X}}{C}, \tag{9}$$

where C is a nonzero constant or another variable's steady-state value that provides a natural scale for X_t.

Remark 1. *The choice of normalizing variable C should be guided by the model's structure. A natural candidate is often the steady-state value of another variable that appears in the same equation and shares the same units as X_t. For instance, if X_t represents net foreign assets (which may be zero in steady state), normalizing by steady-state consumption or output is a common choice.*

6.1 The Sum Rule with a Zero Steady-State Component

Suppose

$$Z_t = a\, X_t + b\, Y_t, \qquad \bar{X} = 0,\; \bar{Z} \neq 0,\; \bar{Y} \neq 0.$$

For Y_t and Z_t we can use the standard percentage deviations y_t and z_t, but for X_t we normalize by some nonzero constant C:

$$\hat{x}_t = \frac{X_t - \bar{X}}{C} = \frac{X_t}{C}.$$

Derivation. Begin with the first-order Taylor expansion:

$$Z_t = \bar{Z} + a(X_t - \bar{X}) + b(Y_t - \bar{Y}).$$

Dividing both sides by $\bar{Z}$:

$$\frac{Z_t - \bar{Z}}{\bar{Z}} = \frac{a(X_t - \bar{X})}{\bar{Z}} + \frac{b(Y_t - \bar{Y})}{\bar{Z}}.$$

Now note that $\bar{X} = 0$ implies $\bar{Z} = b\bar{Y}$. For the X_t term, multiply and divide by C:

$$z_t = \frac{aC}{\bar{Z}}\,\hat{x}_t + \frac{b\bar{Y}}{\bar{Z}}\,y_t.$$

Since $\bar{Z} = b\bar{Y}$, the coefficient on y_t simplifies to one, giving

$$z_t = \frac{aC}{\bar{Z}}\,\hat{x}_t + y_t. \tag{10}$$

□

Note how the structure mirrors the standard sum rule, except that the component with a zero steady state enters through the normalized deviation $\hat{x}_t$ rather than a percentage deviation, and its coefficient depends on the normalizing constant C.

6.2 Variables with Negative Steady-State Values

When $\bar{X} < 0$, the same problem arises: $\ln \bar{X}$ is undefined. The remedy is identical—normalize by a different (positive) quantity:

$$\hat{x}_t \equiv \frac{X_t - \bar{X}}{C}, \qquad C > 0.$$

Everything else proceeds as before: Taylor-expand in levels, then translate into the appropriate hat variables for each component.

6.3 Application: A Linearization from Obstfeld and Rogoff

As a concrete illustration, consider the consumption Euler equation from a sticky-price international RBC model (as in the Chapter 10 Obstfeld and Rogoff's textbook). In that case, the Steady State can change, and we may want to focus on the dynamics of it with respect to a particular steady-state, the symmetric steady state, labeled with the subscript 0.

Thus, we want to log-linearize the following equation around the symmetric steady state:

$$\bar{C} + \delta\bar{B} + \frac{\bar{P}(h)\bar{y}}{\bar{P}}$$

In the symmetric steady-state, however, the net foreign assets are zero: $B_0 = 0$ and thus we have to adjust the log-linearization.

For this, we defined all log-linear variables as usual $\hat{x} = \frac{\bar{X}-X_0}{X_0}$, but we change the definitions of the log-linear bonds: $\hat{b} = \frac{\bar{B}-\cancelto{B_0}{0}}{C_0}$.

From the model we also know that $C_0 = y_0$, and $\frac{P_0(h)}{P_0} = 1$, which we can use to simplify the calculations.[2]

Now we take a Taylor expansion of the target equation around the symmetric steady state:

$$
\begin{aligned}
\bar{C} &= C_0 + \delta(\bar{B} - B_0) + \frac{y_0}{P_0}(\bar{P}(h) - P_0(h)) - \frac{P_0(h)y_0}{P_0^2}(\bar{P} - P_0) + \frac{P_0(h)}{P_0}(\bar{y} - y_0) \\
\bar{C} - C_0 &= \delta(\bar{B} - B_0) + \frac{y_0}{\cancel{P_0}}\cancel{P_0(h)}\frac{(\bar{P}(h) - P_0(h))}{P_0(h)} - \frac{\cancel{P_0(h)}y_0}{\cancel{P_0}}\frac{(\bar{P} - P_0)}{P_0} + \cancel{\frac{P_0(h)}{P_0}}(\bar{y} - y_0) \\
\frac{\bar{C} - C_0}{C_0} &= \delta\frac{(\bar{B} - B_0)}{C_0} + \cancel{\frac{y_0}{C_0}}\frac{(\bar{P}(h) - P_0(h))}{P_0(h)} - \cancel{\frac{y_0}{C_0}}\frac{(\bar{P} - P_0)}{P_0} + \cancel{\frac{\bar{y}}{C_0}}\frac{(\bar{y} - y_0)}{\bar{y}} \\
\hat{c} &= \delta\hat{b} + \hat{p(h)} + \hat{y} - \hat{p}
\end{aligned}
$$

The key lesson is that the linearization mechanics remain the same—Taylor expand, then express in deviation form—but one must be careful about which normalization is applied to each variable.

7 Important Caveats

Two limitations of log-linearization deserve emphasis:

(i) **Accuracy.** All results in this note are valid only up to first order. For questions that depend on second-order effects—such as welfare comparisons across policies—a first-order approximation is generally insufficient, and one should use second-order (or higher) perturbation methods instead.

(ii) **Zero steady states.** As discussed in Section 6, the standard percentage-deviation definition fails when the steady-state value is zero or negative. In such cases, normalizing by an alternative quantity is necessary but introduces an additional modeling choice (the normalizing variable) that should be guided by the economics of the problem.

[2] For clarity, $P(h)$ denote the prices at the home country (i.e., "(h)" is similar to a subscript or label and is not a variable of the model itself).