Econ 5322 - MS Macroeconomics Spring 2025
Granados

Problem set # 2

Answer Key

1. **(RBC model with elastic labor supply)** Consider the RBC model studied in class. Assume that labor supply is variable: Agents choose optimally how much labor to supply in each period. The period utility takes the form:

$$u(C_t, N_t) = \frac{\left[C_t^{\rho}(1-N_t)^{1-\rho}\right]^{1-\gamma}}{1-\gamma}$$

with $\gamma > 0$ and $0 < \rho < 1$ Utility is no longer separable between consumption and leisure (unless we assume $\gamma = 1$). All other assumptions are unchanged (relative to the model studied in class).

   (a) Write the first-order conditions that determine agents' optimal behavior. Explain these first-order conditions intuitively.

   [Hint: The marginal utility of consumption (or leisure) depends on leisure (or consumption) now. In addition to usual intepretations, consider this when comparing to the fixed-labor case.]

(Ans) Given the assumed form of the utility function, the marginal utility of consumption (the derivative of the utility function with respect to consumption) is:

$$u_C(C, 1-N) = \rho C^{\rho-1}(1-N)^{1-\rho}\left[C^{\rho}(1-N)^{1-\rho}\right]^{-\gamma} = \frac{\rho}{C}\left[C^{\rho}(1-N)^{1-\rho}\right]^{1-\gamma}$$

and the marginal utility of leisure (the derivative of the utility function with respect to leisure) is:

$$u_{1-N}(C, 1-N) = (1-\rho)C^{\rho}(1-N)^{-\rho}\left[C^{\rho}(1-N)^{1-\rho}\right]^{-\gamma} = \frac{1-\rho}{1-N}\left[C^{\rho}(1-N)^{1-\rho}\right]^{1-\gamma}$$

The Euler equation for capital accumulation is:

$$u_C(C_t, 1-N_t) = \beta \mathbb{E}_t\left[u_C(C_{t+1}, 1-N_{t+1})R_{t+1}\right]$$

where R_{t+1} is the return to capital at time $t+1$. Hence:

$$\frac{1}{C_t}\left[C_t^{\rho}(1-N_t)^{1-\rho}\right]^{1-\gamma} = \beta \mathbb{E}_t\left\{\frac{1}{C_{t+1}}\left[C_{t+1}^{\rho}(1-N_{t+1})^{1-\rho}\right]^{1-\gamma} R_{t+1}\right\}$$

Intuition: This condition equates the marginal cost of saving and investing into capital accumulation today (marginal utility of forgone consumption) to the discounted expected marginal benefit from the investment tomorrow (discounted expected return to capital evaluated in terms of the marginal utility of consumption that today's investment will generate tomorrow).

The transversality condition is:

$$\lim_{T\to\infty} E_t\left[\beta^T u_C\left(C_{t+T}, 1-N_{t+T}\right) R_{t+T} K_{t+T}\right] = 0$$

or,

$$\lim_{T\to\infty} E_t\left[\beta^T \frac{\rho}{C_{t+T}}\left[C_{t+T}^{\rho}\left(1-N_{t+T}\right)^{1-\rho}\right]^{1-\gamma} R_{t+T} K_{t+T}\right] = 0$$

Intuition: This condition ensures that the agent does not have unused resources "at the end" of his/her lifetime. If the limit were strictly positive, the agent would be better of increasing consumption permanently in all periods by the amount such that the limit is zero (since nothing in the expression can be negative, the limit can never be strictly negative).

The first-order consumption for optimal labor supply is:

$$u_{1-N}(C_t, 1-N_t) = u_C(C_t, 1-N_t) w_t$$

or:

$$\frac{1-\rho}{1-N_t}\left[C_t^{\rho}(1-N_t)^{1-\rho}\right]^{1-\gamma} = \frac{\rho}{C_t}\left[C_t^{\rho}(1-N_t)^{1-\rho}\right]^{1-\gamma} w_t$$

which implies:

$$\frac{1-\rho}{1-N_t} = \frac{\rho}{C_t} w_t$$

or:

$$\frac{C_t}{1-N_t} = \frac{\rho}{1-\rho} w_t$$

Intuition: This condition equates the marginal cost to the agent of supplying an additional unit of labor (the marginal utility of foregone leisure) to the marginal benefit of supplying that labor (the real wage received in payment for it, evaluated in terms of the marginal utility of consumption it allows the agent to buy).

(b) Assume $\gamma = 1$ for the rest of this problem. This implies that the utility function becomes $\rho \log C_t + (1-\rho)\log(1-N_t)$. Solve for the balanced growth path of the model and log-linearize the model around it using the technique described in the lecture.

[Hint: with $\gamma = 1$ you are looking at a special case of the variable-labor model we studied in class. Rewrite the utility function as $\rho\left[\log C_t + (1-\rho)\log(1-N_t)/\rho\right]$ and note that maximizing this is the same as maximizing the original utility function. Compare this to the utility function for the variable-labor model in the slides and you should see that, if you set $\theta = (1-\rho)/\rho$, you have a special case of the slides, which you can follow to solve the problem.]

(Ans) By assuming $\gamma = 1$ the utility becomes logarithmic and the utility function becomes (after using the hint I provided):

$$\rho\left[\log C_t + \frac{1-\rho}{\rho}\log(1-N_t)\right]$$

On the other hand, the slides on the model with variable labor supply assume that the utility function is:

$$\log C_t + \theta \frac{(1-N_t)^{1-\gamma_n}}{1-\gamma_n}$$

If you assume $\gamma_n = 1$ that becomes

$$\log C_t + \theta \log(1-N_t)$$

And note that maximizing $\mathbb{E}_t \sum_{s=t}^{\infty} \beta^{s-t}[\log C_t + \theta \log(1-N_s)]$ is equivalent to maximizing $\rho \mathbb{E}_t \sum_{s=t}^{\infty} \beta^{s-t} \left[\log C_t + \frac{1-\rho}{\rho} \log(1-N_s)\right]$ if you set $\theta = (1-\rho)/\rho$. Thus, you can just set $\gamma_n = 1$ and $\theta = (1-\rho)/\rho$ in the slides material on the model with labor supply (Topic 2, slides 71-88 and Appendix E-G) to solve the rest of this problem.

Still, for your convenience, some of the material is summarized below:

Once you assume $\gamma = 1$ and utility becomes $\rho \left[\log C_t + \frac{1-\rho}{\rho} \log(1-N_t)\right]$ the Balance Growth path you will obtain is:

$$\frac{\bar{A}_t}{\bar{K}_t} \approx \frac{1}{N}\left(\frac{r+\delta}{1-\alpha}\right)^{\frac{1}{\alpha}}$$
$$\frac{\bar{Y}_t}{\bar{K}_t} \approx \frac{r+\delta}{1-\alpha}$$
$$\frac{\bar{C}_t}{\bar{K}_t} \approx \frac{r+\alpha\delta - g(1-\alpha)}{1-\alpha}$$

For a detailed derivation, refer to the slides (Topic 2, 116-118)

Log-linearized model:

Depart from the FOC conditions in (a). The log-linear version of $\frac{1-\rho}{\rho}(1-N_t)^{-1} = C_t^{-1} w_t$ is:

$$n_t \approx \mu\left[(1-\alpha)k_t + \alpha a_t - c_t\right]$$

For a detailed derivation, refer to lecture slides pages 78-79 (note that $\theta = (1-\rho)/\rho$ and $w_t = A_t^{\alpha}\left(\frac{k_t}{N_t}\right)^{1-\alpha}$ in equation (40) in the slides.)

The law of motion for capital:

We have that $C_t + K_{t+1} = (1-\delta)K_t + Y_t$ and (production function in log-linear form) $y_t + \alpha(a_t + n_t) + (1-\alpha)k_t$, which imply a log-linear version of the capital accumulation equation given by:

$$k_{t+1} = \lambda_1 k_t + \lambda_2(a_t + n_t) + (1-\lambda_1-\lambda_2)c_t$$

Gross return rate to capital accumulation:

In levels the gross return is $R_{t+1} = (1-\alpha)\left(\frac{A_{t+1}N_{t+1}}{K_{t+1}}\right)^{\alpha} + 1 - \delta$, and the log-linear version of the interest rate equation is:

$$r_{t+1} = \lambda_3(a_{t+1} + n_{t+1} - k_{t+1})$$

Euler Equation:

The Euler Equation in levels is simplified relative to what found in (a) given our additional assumption and is given by $C_t^{-1} = \beta \mathbb{E}_t \left(C_{t+1}^{-1} R_{t+1} \right)$

The log-linear version of the Euler Equation is:

$$\mathbb{E}_t \left(c_{t+1} - c_t \right) = \lambda_3(a_{t+1} + n_{t+1} - k_{t+1})$$

For details on this equation see the slide 77 (and notice we used the solution for r_{t+1}).

(c) Use the "guess and check" procedure shown in class and the method of undetermined coefficients to solve the model

(Ans) The "guess and check" approach to solving the model is the approach that consists on guessing that the solution takes the form in equation (46) in the slide 81, i.e., that the solution has endogenous variables as linear functions of the smallest set of state variables for the model (capital at the start of the period and technology during the period; a guess that is safe for us to make when models have a unique solution) and then substituting the guessed solution into the equation of the model, using the method of undetermined coefficients, to find how the η coefficients in the solution depend on the parameters of the model.

As mentioned, every detail you may need is in the slides. In particular for finding the solution, you can follow the slides 121-125 (Appendix G)

2. **(RBC model with government spending shocks)** Consider again the stochastic growth model, focus on the fixed-labor case, but now allow for government spending shocks as a source of fluctuations.

The representative consumer maximizes:

$$\mathbb{E}_t \sum_{i=0}^{\infty} \beta^i \frac{C_{t+i}^{1-\gamma}}{1-\gamma}$$

where $0 < \beta < 1$ and $\gamma > 0$.

The law of motion of capital is:

$$K_{t+1} = (1-\delta)K_t + Y_t - C_t - X_t \qquad (*)$$

X denotes exogenous government spending, financed through lump-sum taxation.

Output is:

$$Y_t = A_t^{\alpha} K_t^{1-\alpha}, \quad \text{with } 0 < \alpha < 1$$

(a) Obtain the Euler Equation for capital accumulation. Explain the intuition.

(Ans)

$$\max_{C_t, K_{t+1}} E_t \sum_{i=0}^{\infty} \beta^i \frac{C_{t+i}^{1-\gamma}}{1-\gamma}$$

$$\text{s.t.} \quad C_t + K_{t+1} = (1-\delta) K_t + Y_t + X_t$$

$$Y_t = A_t^{\alpha} K_t^{1-\alpha}$$

We can replace Y_t into the budget constraint and obtain from it an expression for C_t:

$$C_t = (1-\delta) K_t - K_{t+1} + A_t^{\alpha} K_t^{1-\alpha} - X_t$$

With this expression (valid for any period $t+i$), we replace C_t into the objective function:

$$\max_{K_{t+1}} \mathbb{E}_t \sum_{i=0}^{\infty} \beta^i \frac{\left[(1-\delta) K_t - K_{t+1} + A_t^{\alpha} K_t^{1-\alpha} - X_t\right]^{1-\gamma}}{1-\gamma}$$

F.O.C with respect to K_{t+1}:

$$\mathbb{E}_t \left\{ - \left[(1-\delta) K_t - K_{t+1} + A_t^{\alpha} K_t^{1-\alpha} + X_t\right]^{-\gamma} \right.$$
$$\left. + \beta \left(1 - \delta + (1-\alpha) A_{t+1}^{\alpha} K_{t+1}^{-\alpha}\right) \left[(1-\delta) K_{t+1} - K_{t+2} + A_{t+1}^{\alpha} K_{t+1}^{1-\alpha} + X_{t+1}\right]^{-\gamma} \right\} = 0$$

We can substitute back consumption in each period and remove the expectation operator for information already known at t:

$$C_t^{-\gamma} = \beta \mathbb{E}_t \left(C_{t+1}^{-\gamma} \left(1 + (1-\alpha) A_{t+1}^{\alpha} K_{t+1}^{-\alpha} - \delta\right)\right)$$

Now, we can replace $r_{t+1} = \frac{\partial Y_{t+1}}{\partial K_{t+1}} = A_{t+1}^{\alpha} K_{t+1}^{-\alpha}$ to get an analogous expression to the one obtained in the lecture slides:

$$C_t^{-\gamma} = \beta \mathbb{E}_t \left(C_{t+1}^{-\gamma} \left(1 + r_{t+1} - \delta\right)\right)$$

Intuition: in an optimal allocation (equilibrium), a household should be indiferent between increasing consumption with one extra dollar, or saving it, so it generates a return and consumption can be made in the future. Thus, the marginal utility of consumption today should be equal to the discounted gross return of the savings evaluated in terms of marginal utility of consumption tomorrow.

(b) Solve for the balanced growth path (here it's useful to treat $\bar{X}_t / \bar{Y}_t$ as an exogenous variable). Assume $\bar{A}_{t+1}/\bar{A}_t = \bar{X}_{t+1}/\bar{X}_t = G$.

(Ans)

$$Y_t = A_t^{\alpha} K_t^{1-\alpha}$$
$$R_{t+1} = (1-\alpha)\left(\frac{A_{t+1}}{K_{t+1}}\right)^{\alpha} + 1 - \delta$$

In steady state, the gross return is R, and the Euler Equation becomes $G^{\gamma} = \beta R$, rearranging the formula for $\bar{R}_{t+1}$:

$$\frac{\bar{A}_t}{\bar{K}_t} = \left[\frac{\frac{G^{\gamma}}{\beta} - (1-\delta)}{1-\alpha}\right]^{1/\alpha} \approx \left(\frac{r+\delta}{1-\alpha}\right)^{1/\alpha}$$
$$\frac{\bar{Y}_t}{\bar{K}_t} = \left(\frac{\bar{A}_t}{\bar{\bar{K}}_t}\right)^{\alpha} \approx \frac{r+\delta}{1-\alpha}$$

Now, consider and rearrange the budget constraint (or capital dynamics formula) as follows:

$$K_{t+1} = (1-\delta)K_t + Y_t - C_t - X_t$$
$$\frac{\bar{K}_{t+1}}{\bar{K}_t}\frac{\bar{K}_t}{\bar{Y}_t} = (1-\delta)\frac{\bar{K}_t}{\bar{Y}_t} + \frac{\bar{Y}_t}{\bar{Y}_t} - \frac{\bar{C}_t}{\bar{Y}_t} - \frac{\bar{X}_t}{\bar{Y}_t}$$
$$(1+g)\left(\frac{1-\alpha}{r+\delta}\right) = (1-\delta)\left(\frac{1-\alpha}{r+\delta}\right) + 1 - \frac{\bar{C}_t}{\bar{Y}_t} - \frac{\bar{X}_t}{\bar{Y}_t}$$
$$\frac{\bar{C}_t}{\bar{Y}_t} = 1 - (\delta+g)\left(\frac{1-\alpha}{r+\delta}\right) - \frac{\bar{X}_t}{\bar{Y}_t}$$

We use these results to obtain an expression for $\frac{\bar{C}_t}{\bar{K}_t}$:

$$\frac{\bar{C}_t}{\bar{K}_t} = \frac{\bar{C}_t}{\bar{Y}_t} * \frac{\bar{Y}_t}{\bar{K}_t} = \left(1 - (\delta+g)\left(\frac{1-\alpha}{r+\delta}\right) - \frac{\bar{X}_t}{\bar{Y}_t}\right)\left(\frac{r+\delta}{1-\alpha}\right) = \frac{r+\delta}{1-\alpha}\left(1 - \frac{\bar{X}_t}{\bar{Y}_t}\right) - (\delta+g)$$

Finally, from the capital dynamics formula we now can get an expression for $\frac{\bar{X}_t}{\bar{K}_t}$

$$\frac{\bar{K}_{t+1}}{\bar{K}_t} = (1-\delta)\frac{\bar{K}_t}{\bar{K}_t} + \frac{\bar{Y}_t}{\bar{K}_t} - \frac{\bar{C}_t}{\bar{K}_t} - \frac{\bar{X}_t}{\bar{K}_t}$$
$$(1+g) = (1-\delta) + \left(\frac{r+\delta}{1-\alpha}\right) - \left(1 - (\delta+g)\left(\frac{1-\alpha}{r+\delta}\right) - \frac{\bar{X}_t}{\bar{Y}_t}\right)\left(\frac{r+\delta}{1-\alpha}\right) - \frac{\bar{X}_t}{\bar{K}_t}$$
$$\frac{\bar{X}_t}{\bar{K}_t} = \frac{\bar{X}_t}{\bar{Y}_t}\left(\frac{r+\delta}{1-\alpha}\right)$$

Notice this derivation was simpler given we considered $\frac{\bar{X}_t}{\bar{Y}_t}$ to be an exogenous quantity.

(c) Assume log-normality, so that $\log(\mathbb{E}_t X_{t+1}) \approx \mathbb{E}_t(\log X_{t+1}) + \frac{1}{2}Var_t(\log X_{t+1}))$ for any variable X, and homoskedasticity (variances and covariances are constant).

Log-linearize equation (*), the Euler Equation, and the expression for the gross return to capital accumulation around the steady state.

(Ans) The law of motion for capital:

$$K_{t+1} = (1-\delta)K_t + Y_t - C_t - X_t$$
$$y_t = \alpha a_t + (1-\alpha)k_t$$
$$\bar{K}_{t+1}\frac{dK_{t+1}}{\bar{K}_{t+1}} = (1-\delta)\bar{K}_t\frac{dK_t}{\bar{K}_t} + \bar{Y}_t\frac{dY_t}{\bar{Y}_t} - \bar{C}_t\frac{dC_t}{\bar{C}_t} - \bar{X}_t\frac{dX_t}{\bar{X}_t}$$

rearranging, substituting the steady-state ratios (derived before) and expressing the variables in deviations of the steady state:

$$k_{t+1} = \left(\frac{1+r}{1+g}\right)k_t + \frac{\alpha(r+\delta)}{(1-\alpha)(1+g)}a_t - \frac{1}{1+g}\left(\frac{r+\delta}{1-\alpha}\left(1-\frac{\bar{X}_t}{\bar{Y}_t}\right) - (\delta+g)\right)c_t$$
$$-\frac{\bar{X}_t}{\bar{Y}_t}\left(\frac{r+\delta}{(1-\alpha)(1+g)}\right)x_t$$

Euler Equation:

$$C_t^{-\gamma} = \beta\mathbb{E}_t\left(C_{t+1}^{-\gamma}R_{t+1}\right)$$

The log-linear version of this equation is (refer to Topic 2, slides 43-46 for a step-by-step derivation)

$$\mathbb{E}_t\left(c_{t+1}-c_t\right) = \frac{1}{\gamma}\mathbb{E}_t r_{t+1}$$

Gross return rate to capital accumulation:

$$R_{t+1} = (1-\alpha)\left(\frac{A_{t+1}}{K_{t+1}}\right)^{\alpha} + 1 - \delta$$

The log-linear version of the intrest rate equation is:

$$r_{t+1} \approx \frac{\alpha(r+\delta)}{1+r}(a_{t+1}-k_{t+1})$$

For details on this derivation, refer to slides 47-49

(d) Assume $a_t = 0$ for all t (i.e., there are no percentage deviations of technology from the steady state). Assume $x_t = \phi x_{t-1} + \varepsilon_t$, $\mathbb{E}_{t-1}\varepsilon_t = 0$. Show that the model reduces to:

$$k_{t+1} = \lambda_1 k_t + \lambda_4 x_t + (1-\lambda_1-\lambda_2-\lambda_4)c_t$$
$$\mathbb{E}_t(c_{t+1}-c_t) = -\frac{\lambda_3}{\gamma}k_{t+1}$$
$$x_t = \phi x_{t-1} + \varepsilon_t$$

with $\lambda_1 \equiv \frac{1+r}{1+g}$, $\lambda_2 \equiv \frac{\alpha(r+\delta)}{(1-\alpha)(1+g)}$, $\lambda_3 \equiv \frac{\alpha(r+\delta)}{1+r}$, $\lambda_4 = -\frac{(r+\delta)\bar{X}_t/\bar{Y}_t}{(1-\alpha)(1+g)}$, and $\mathbb{E}_{t-1}\varepsilon_t = 0$.

(Ans) We had from before,

$$k_{t+1} = \left(\frac{1+r}{1+g}\right) k_t + \frac{\alpha(r+\delta)}{(1-\alpha)(1+g)} a_t - \frac{1}{1+g}\left[\frac{r+\delta}{1-\alpha} - (\delta+g) - \frac{\bar{X}_t}{\bar{Y}_t}\left(\frac{r+\delta}{1-\alpha}\right)\right] c_t$$
$$-\frac{\bar{X}_t}{\bar{Y}_t}\left(\frac{r+\delta}{(1-\alpha)(1+g)}\right) x_t$$

replacing $a_t = 0$ in this expression and in the Euler equation:

$$k_{t+1} = \lambda_1 k_t + (1-\lambda_1-\lambda_2-\lambda_4)\, c_t + \lambda_4 x_t$$
$$E_t\left(c_{t+1} - c_t\right) = \frac{1}{\gamma} E_t r_{t+1} = \frac{\alpha(r+\delta)}{1+r} E_t\left(a_{t+1} - k_{t+1}\right) = -\frac{\lambda_3}{\gamma} k_{t+1}$$

Thus, the model is reduced to:

$$k_{t+1} = \lambda_1 k_t + \lambda_4 x_t + (1-\lambda_1-\lambda_2-\lambda_4) c_t$$
$$\mathbb{E}_t(c_{t+1} - c_t) = -\frac{\lambda_3}{\gamma} k_{t+1}$$
$$x_t = \phi x_{t-1} + \varepsilon_t$$

Intuition: upon increases in the government spending, households expect that future taxes will increase and lead to a reduction in their future income; thus, agents decide to consume less today and save more to pay future taxes. The higher government spending can also change the investment level. Therefore, a government shock can affect the path of consumption and investment.

(e) The solution for consumption and capital has the form:

$$c_t = \eta_{ck} k_t + \eta_{cx} x_t$$
$$k_{t+1} = \eta_{kk} k_t + \eta_{kx} x_t$$

What is the intuition for this solution? Briefly describe how these equations and $x_t = \phi x_{t-1} + \varepsilon_t$ can be used to trace the response of capital and consumption to a government spending shock.

3. Briefly: Describe the possible calibration approaches to evaluate the empirical performance of models.

(Ans) As we have the benchmark values of g, r, α, δ and $\frac{\bar{X}_t}{\bar{Y}_t}, \lambda_1, \lambda_2, \lambda_3$ and λ_4 are known, we can search for values of ϕ and σ (persistence and variance of shocks to government spending) such that the empirical moments generated by the model (e.g., mean values of variables such as consumption to gdp ratio, variances, correlations among economic variables, etc.) are consistent with, and thus "match", those observed in the data (see page 53 in Topic 2 slides for further comments on models calibration).